LINEARISATION OF MULTIWELL ENERGIES

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Abstract. Linear elasticity can be rigorously derived from finite elasticity under the assumption of small loadings in terms of Gamma-convergence. This was first done in the case of one-well energies with super-quadratic growth and later generalised to different settings, in particular to the case of multi-well energies where the distance between the wells is very small (comparable to the size of the load). In this paper we study the case when the distance between the wells is independent of the size of the load. In this context linear elasticity can be derived by adding to the multi-well energy a singular higher order term which penalises jumps from one well to another. The size of the singular term has to satisfy certain scaling assumptions whose optimality is shown in certain regimes. Finally, the derivation of linear elasticty from a two-well discrete model is provided, showing that the role of the singular perturbation term is played in this setting by interactions beyond nearest neighbours.

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Contents

Introduction

Consider a hyperelastic body whose reference configuration Ω is a bounded domain of $\mathbb{R}^d$. The stored energy associated with a deformation $v : \Omega \to \mathbb{R}^d$ can be written as

$$\int_\Omega W(x, \nabla v)\,\mathrm{d}x\,,$$

where $W \colon \Omega \times \mathbb{R}^{d\times d} \to [0, +\infty]$ is the energy density encoding the physical and mechanical properties of the material under consideration. If the reference configuration is stress-free, we may assume that the identity matrix I minimises W and, without loss of generality, that $W(x, I) \equiv 0$. By frame-indifference, we may also assume that $W(x, RA) = W(x, A)$ for every $A \in \mathbb{R}^{d\times d}$ and every $R \in SO(d)$, hence $W(x, \cdot)$ is null on $SO(d)$. If an external load $l(x)$ is applied, the total energy of the body is

$$\int_\Omega W(x, \nabla v)\,\mathrm{d}x - \int_\Omega lv\,\mathrm{d}x\,.$$

The derivation of a linear elastic theory from nonlinear elasticity is based on the observation that, since the reference configuration is an equilibrium when no external load is applied, it is natural to expect that small external loads will produce small displacements.

It is then convenient to introduce a small parameter $\varepsilon > 0$ and write the load as εl. Then, writing the deformation in terms of the scaled displacement $\varepsilon u(x)$, that is $v(x) = x + \varepsilon u(x)$, the associated energy can be written, up to an additive constant, as

$$\int_\Omega W(x, I + \varepsilon\nabla u)\,\mathrm{d}x - \varepsilon^2 \int_\Omega lu\,\mathrm{d}x\,. \tag{0.1}$$

Assuming that $W(x, \cdot)$ is C^2 in a neighbourhood of I and that ∇u is bounded, a Taylor expansion of $W(x, I + \varepsilon\nabla u(x))$ about I yields that the energy, scaled by $1/\varepsilon^2$, converges as $\varepsilon \to 0$ to

$$\int_\Omega D^2W(x, I)[\nabla u]^2\,\mathrm{d}x - \int_\Omega lu\,\mathrm{d}x\,, \tag{0.2}$$

where $D^2W(x, I)[A]^2$ denotes the second derivative of W with respect to the second variable evaluated at the point (x, I) and applied to the pair $[A, A]$. By frame-indifference, the quadratic form $A \mapsto D^2W(x, I)[A]^2$ depends only on $(A^T + A)/2$, the symmetric part of A, hence

$$\int_\Omega D^2W(x, I)[\nabla u]^2\,\mathrm{d}x = \int_\Omega D^2W(x, I)[e(u)]^2\,\mathrm{d}x\,, \tag{0.3}$$

where $e(u) := (\nabla u^T + \nabla u))/2$ is the symmetric part of the displacement gradient ∇u. The functional (0.3) represents the linear elastic energy associated with the displacement u.

The above formal derivation of linear elasticity can be made rigorous in terms of convergence of absolute minimisers of boundary value problems associated with the functionals (0.1) and (0.2). This has been done in the framework of Γ-convergence in [9, 16, 2, 1]. In the seminal paper [9], convergence of minimisers was proven in $H^1(\Omega; \mathbb{R}^d)$ under the condition

$$W(x, A) \geq C\,\mathrm{dist}^2(A, SO(d))\,. \tag{0.4}$$

This assumption was later replaced in [2] by the weaker assumption that (0.4) holds only in a neighbourhood of $SO(d)$, while away from such neighbourhood W has a growth slower than quadratic, specifically,

$$W(x, A) \geq C\,\mathrm{dist}^q(A, SO(d))\,, \quad 1 < q < 2\,.$$

In this case, the convergence of minimisers holds in $W^{1,q}(\Omega;\mathbb{R}^d)$. Further generalisations of these results have been obtained in [16, 1] in the case of multi-well energies where the distance between the wells is of order ε. It is worth mentioning that, in the context of Γ-convergence, linear elastic energies have been also obtained as macroscopic limit of nonlinear atomistic energies in [5, 17].

In the proof of all the aforementioned results, a delicate step is to show compactness properties (in some Sobolev space) of sequences of admissible displacement fields u_ε whose energy is uniformly bounded in ε. To this end, a fundamental tool turns out to be the well-known rigidity estimate of Friesecke, James, and Müller [12] (see Theorem 2.8). Indeed, if we assume for example that W satisfies (0.4), such estimate implies that there exist rotations $R_\varepsilon \in SO(d)$ such that

$$\int_\Omega |I + \varepsilon\nabla u_\varepsilon - R_\varepsilon|^2 \,\mathrm{d}x \le C\varepsilon^2 .$$

Then, assuming that the boundary data are (a perturbation of) the identity, one can show that R_ε lies in an ε-neighbourhood of I and finally that

$$\int_\Omega |\nabla u_\varepsilon|^2 \,\mathrm{d}x \le C .$$

The purpose of this paper is to investigate the case when the zero level set of the energy density consists of several wells whose mutual distance is independent of ε. The derivation of linear elasticity in this setting is relevant, since energies of this form naturally arise in many models, for example in the gradient theory of solid-solid phase transitions. We point out that, if one wishes to follow the strategy outlined above, a suitable version of the rigidity estimate will be needed. However, such an estimate is known to hold only when the wells are strongly incompatible (see e.g. [6, 10] and Remark 2.3). If instead we assume that the wells have a fixed distance and are rank-one connected, in contrast to the case when the distance vanishes, we cannot expect compactness in any Sobolev space. Indeed, if we assume for example that $W(x,\cdot) \equiv 0$ on $SO(d) \cup SO(d)U$, with $U \in \mathbb{R}^{d\times d}$ and $\mathrm{rank}(U - I) = 1$, then one can define a continuous deformation v such that $\nabla v = I$ in Ω_1 and $\nabla v = U$ in Ω_2, where $\{\Omega_1, \Omega_2\}$ is a partition of Ω into sets of positive Lebesgue measure. Thus, the corresponding displacements $u_\varepsilon(x) = (v(x) - x)/\varepsilon$ have zero energy and satisfy $u_\varepsilon = 0$ on $\partial\Omega_1\setminus\Omega$; moreover, $\nabla u_\varepsilon = (U - I)/\varepsilon$ in Ω_2, which implies that $\|\nabla u_\varepsilon\|_{L^r}$ diverges as $\varepsilon \to 0$, for every $r \ge 1$.

In order to recover compactness, the idea is then to add to the energy a higher order singular perturbation that penalises the transitions between the wells. Specifically, we introduce an additional small parameter η and assume that the stored energy associated with a deformation v is of the form

$$E^\eta(v) = \int_\Omega W(x,\nabla v)\,\mathrm{d}x + \eta^2 \int_\Omega |\nabla^2 v|^2 \,\mathrm{d}x , \quad v \in H^2(\Omega;\mathbb{R}^d) . \tag{0.5}$$

The asymptotic behaviour as $\eta \to 0$ of functionals of this type, in particular in the case where the order parameter is not constrained to be curl-free, has been object of many papers; among these, we refer to Modica and Mortola [15] where a Γ-convergence analysis has been first performed in the scalar case. The analysis of the curl-free case is much more complex. A rigorous result in terms of Γ-convergence was obtained in [7], in dimension two, for two rank-one connected wells of the form $K_1 = SO(d)$ and $K_2 = SO(d)U$, $U \in GL^+(\mathbb{R}^d)$: the Γ-limit of the scaled functionals $\eta^{-1}F^\eta$ is finite on functions v such that $\nabla v \in BV(\Omega; K_1 \cup K_2)$ and it is represented by an interfacial energy of the form

$$\int_{J_{\nabla v}} \varphi(\nu_{\nabla v})\,\mathrm{d}\mathcal{H}^1 ,$$

where $J_{\nabla v}$ is the jump set of ∇v and $\nu_{\nabla v}$ is the unit normal to $J_{\nabla v}$. The problem in higher dimension and for more general multiple wells is still open. Nevertheless it can be shown that $E^\eta(v)$ scales like $\eta\,\mathcal{H}^{d-1}(J_{\nabla v})$. In view of this, one can look for a suitable scaling of $\eta = \eta(\varepsilon)$ with respect to ε which guarantees compactness for the scaled energies $\varepsilon^{-2}E^\eta(x+\varepsilon u_\varepsilon)$.

Let us illustrate a heuristic argument in the case discussed above with two wells K_1 and K_2. Let v_ε be a sequence such that $\varepsilon^{-2}E^\eta(v_\varepsilon)$ is uniformly bounded. As already observed, in the set Ω^*_ε where the deformation v_ε takes values far from K_1, the corresponding scaled displacement u_ε is such that $\nabla u_\varepsilon \sim 1/\varepsilon$. Hence, in order to obtain the boundedness of ∇u_ε, e.g. in L^2, we need that $|\Omega^*_\varepsilon| \le C\varepsilon^2$. From the boundedness of the energy we deduce

$$\mathcal{H}^{d-1}(J_{\nabla v_\varepsilon}) \le C\,\frac{\varepsilon^2}{\eta}\,,$$

and thus, by the isoperimetric inequality,

$$\min\Big\{\big|\Omega^*_\varepsilon\big|,\ \big|\Omega\backslash\Omega^*_\varepsilon\big|\Big\} \le C\left(\frac{\varepsilon^2}{\eta}\right)^{d/d-1}.$$

If $\eta \ge C\varepsilon^{2/d}$ and if the boundary conditions are such that the minimum above is attained by $\big|\Omega^*_\varepsilon\big|$, we infer that

$$\big|\Omega^*_\varepsilon\big| \le C\varepsilon^2\,,$$

which in turn yields compactness. On the other hand, the condition that $\eta \to 0$ as $\varepsilon \to 0$ ensures that the singular perturbation acts only as a penalisation and does not contribute explicitly to the limit functional, namely, if u is a fixed displacement with ∇u bounded,

$$\lim_{\varepsilon\to 0}\frac{1}{\varepsilon^2}E^\eta(x+\varepsilon u) = \int_\Omega D^2W(x,I)[e(u)]^2\,\mathrm{d}x\,.$$

We make rigorous this argument by showing that, under the assumption that $C\varepsilon^{2/d} \le \eta(\varepsilon) \ll 1$, sequences of admissible fields with uniformly bounded energies are bounded in L^2, and that the sequence of functionals $\varepsilon^{-2}E^\eta(v_\varepsilon)$ Γ-convergences of to the functional (0.3). More in general we consider functionals of the form (0.5) where the exponent 2 in the singular perturbation term is replaced by any $p > 1$ and where the potential W has an arbitrary but finite number of wells, it grows quadratically close to the wells, and with a power $q \in (0,2]$ away from the wells. The case $q \in (0,1]$ was not covered in [2] and is treated for the first time in the present paper. We obtain compactness properties of the scaled energies in $W^{1,r}$, for some $r \in (1,2]$ depending on p and q, under a suitable scaling of $\eta(\varepsilon)$ which in turn depends on all these parameters (see Theorems 1.6, 1.7 and Tables 1, 2). The Γ-convergence result in this general setting is stated in Theorem 1.8. Moreover we provide examples showing that the compactness results above may not hold if the scaling assumptions are not satisfied and that in most of the cases such scalings are optimal (see Section 3). We point out that all our results are independent of the existence of rank-one connections between the wells as well as of the sign of the determinant of the matrices in the wells.

A fundamental step in our analysis, whose proof is essentially based on the heuristic argument above, is Theorem 2.2, asserting that a suitable power of the energies is controlled from below by the L^r-distance of the deformation gradients from a single well. Moreover, as a technical fact, we remark that in the proof of the Γ-convergence result we have to follow a slightly different strategy than that used in the proof of the analogous result in [2, 9] for one-well potentials, since in our case we get a priori estimates that are in general weaker than those obtained in their analysis (see Section 2.5). Nevertheless our proof could be adopted also in those cases.

The convergence of boundary value problems is stated in Theorems 1.4 and 1.5, where we distinguish the case of zero external load or of a general external load. Indeed in the latter case, in contrast to the one-well model, the uniform boundedness of the total energy for a given sequence u_ε does not in general yield uniform boundedness of $\varepsilon^{-2}E^\eta(x+\varepsilon u_\varepsilon)$ and thus compactness of u_ε in $W^{1,r}$. This can be recovered for a suitable choice of the scaling of $\eta(\varepsilon)$ which in some cases is more restrictive than that needed in the case of zero external load. Finally, analogously to the one-well case, the strong convergence in $W^{1,r}$ of the minimisers is obtained.

The last section of the paper is devoted to the derivation of linear elasticity from a two-well discrete model. Our aim is to show that the role of the singular term in the continuum model is played in this setting by interactions beyond nearest neighbours, which prevent too many jumps from one well to another (see also [4]). We focus on the simple but meaningful case of a two-dimensional discrete system governed by pairwise harmonic interactions between nearest and next-to-nearest neighbours and on a scaling regime that ensures compactness properties of the displacement fields in the weak topology of H^1. The extension of this analysis to a broader class of interacting potentials and to more general scaling regimes goes beyond the purposes of the present paper and will be the object of future work.

Notation. Throughout the paper, the letter C will be used to denote various positive constants, whose precise value may change from line to line; its dependence on other variables will be emphasised only if needed.

The positive (resp., negative) part of x is denoted by x^+ (resp., x^-), while its integer part is denoted by $[x]$. The operator $\wedge$ (resp., $\vee$) denotes the minimum (resp., the maximum) of two numbers.

Given $1 \le p < d$, p^* is the Sobolev exponent defined for $p < d$ by

$$p^* = pd/(d-p)\,.$$

For every $s \ge 1^*$,

$$s_* := ds/(d+s)$$

is the number such that its Sobolev exponent is by $(s_*)^* = s$.

1. Statement of the problem and main result

We consider an elastic body whose reference configuration $\Omega \subset \mathbb{R}^d$ is a bounded, open, and connected set with Lipschitz boundary. For the sake of generality we consider an arbitrary $d \ge 1$, the physically relevant case being $d = 3$.

Let $l \in \mathbb{N}$, $l \ge 2$, let $U_1, \dots, U_l$ be invertible matrices in $\mathbb{R}^{d\times d}$, with $U_1 = I$ the identity matrix, and set

$$K := \bigcup_{i=1}^{l} K_i\,, \quad K_i := SO(d)U_i\,.$$

We assume that the sets K_i, $i = 1, \dots, l$, are all disjoint, namely, that $U_iU_j^{-1} \notin SO(d)$ for each $i \ne j$. Let $W\colon \Omega \times \mathbb{R}^{d\times d} \to [0, +\infty]$ be $(\mathcal{L}\times\mathcal{B})$-measurable, $\mathcal{L}$ and $\mathcal{B}$ denoting the σ-algebras of the Lebesgue measurable subsets of $\mathbb{R}^d$ and the Borel measurable subsets of $\mathbb{R}^{d\times d}$, respectively. We suppose that W satisfies the following properties for a.e. $x \in \Omega$:

(i) $W(x,\cdot)$ is frame indifferent, i.e., $W(x, RF) = W(x, F)$ for all $F \in \mathbb{R}^{d\times d}$ and $R \in SO(d)$;
(ii) $W(x, F) = 0$ if $F \in K$;
(iii) there exists $\sigma > 0$ such that $W(x,\cdot)$ is of class C^2 in $I_\sigma := \{F \in \mathbb{R}^{d\times d}\colon \mathrm{dist}(F, K) < \sigma\}$ and the second derivatives are bounded by a constant independent of x;

(iv) there exists $q \in [0,2]$ such that $W(x,F) \geq C f_q(\mathrm{dist}(F,K))$, where $f_q(t) := t^2 \wedge t^q$ for $t \geq 0$.

In what follows, given $F, A \in \mathbb{R}^{d\times d}$, $D^2W(x,F)[A]^2$ denotes the second derivative of W with respect to F evaluated at the point (x,F) and applied to the pair $[A,A]$, i.e.,

$$D^2W(x,F)[A]^2 = D^2W(x,F)[A,A] = D^2W(x,F)A : A\,,$$

where : is the scalar product between matrices. By frame indifference, we have

$$D^2W(x,I)[A]^2 = D^2W(x,I)[A_{sym}]^2 \quad \text{for every } A \in \mathbb{R}^{d\times d}, \text{ for a.e. } x \in \Omega\,,$$

where A_{sym} denotes the symmetric part of A. Together with assumption (iv), this implies that the quadratic form $D^2W(x,I)[\cdot]^2$ is null on skew matrices and satisfies the coercivity condition

$$D^2W(x,I)[A_{sym}]^2 \geq \lambda |A_{sym}|^2 \quad \text{for every } A \in \mathbb{R}^{d\times d}\,, \text{ for a.e. } x \in \Omega\,, \tag{1.1}$$

for some $\lambda > 0$.

Fixed $p > 1$, we consider the family of singularly perturbed energy functionals, depending on a small parameter $\varepsilon > 0$, defined as

$$E_\varepsilon(v) := \int_\Omega W(x,\nabla v)\,\mathrm{d}x + \eta^p(\varepsilon)\int_\Omega |\nabla^2 v|^p\,\mathrm{d}x\,, \quad v \in W^{2,p}(\Omega;\mathbb{R}^d)\,,$$

where $\eta(\varepsilon) > 0$ satisfies some scaling assumptions which will be specified later on.

We prescribe a Dirichlet condition of the form

$$v(x) = x + \varepsilon\, g(x) \quad \mathcal{H}^{d-1}\text{-a.e. on } \Gamma\,, \tag{1.2}$$

where $g \in W^{1,\infty}(\Omega;\mathbb{R}^d) \cap W^{2,p}(\Omega;\mathbb{R}^d)$, Γ is a nonempty subset of $\partial\Omega$, open in the relative topology, and the values of v and g on Γ are meant in a suitable sense of traces to be made precise below. For technical reasons we require that

$$\mathrm{cap}(\overline{\Gamma}\backslash\Gamma) = 0\,, \tag{1.3}$$

where the definition of capacity is recalled in the Appendix, and that

$$\forall\, i \neq 1\,,\ \forall\, Q \in SO(d)\,,\ \forall\, x \in \mathbb{R}^d\,,\ \mathcal{H}^{d-1}\Big(\Gamma\backslash\big(x + \ker(QU_i - I)\big)\Big) > 0\,. \tag{1.4}$$

The regularity of g ensures that $E_\varepsilon(x+\varepsilon\, g)$ is finite for ε sufficiently small. In terms of the displacement

$$u(x) := \frac{v(x) - x}{\varepsilon}\,,$$

the Dirichlet boundary condition reads as

$$u = g \quad \mathcal{H}^{d-1}\text{-a.e. on } \Gamma\,. \tag{1.5}$$

For future convenience we define for every $s \geq 1$ the sets

$$W^{1,s}_{g,\Gamma}(\Omega;\mathbb{R}^d) := \{u \in W^{1,s}(\Omega;\mathbb{R}^d) \colon u = g \ \ \mathcal{H}^{d-1}\text{-a.e. on } \Gamma\}\,,$$

where the equality holds in the sense of the traces of $W^{1,s}(\Omega;\mathbb{R}^d)$ on Γ. For $s = 2$ we employ the notation

$$H^1_{g,\Gamma}(\Omega;\mathbb{R}^d) := W^{1,2}_{g,\Gamma}(\Omega;\mathbb{R}^d)\,.$$

We set

$$\mathcal{W}^{p,q}_g(\Omega;\mathbb{R}^d) := W^{1,q\vee p}_{g,\Gamma}(\Omega;\mathbb{R}^d) \cap W^{2,p}(\Omega;\mathbb{R}^d)\,.$$

Remark 1.1. *Condition* (1.3) *is a regularity assumption on* Γ*, which is satisfied e.g. when* $\partial\Omega$ *is a* C^1 *manifold and* Γ *is a* C^1 *submanifold with* $\partial\Gamma$ *of class* C^1*. It will be used in the proof of Proposition A.3. However,* (1.3) *is not needed if the exponent* p *defined above is sufficiently large and the boundary condition is slightly stronger (Remark A.5).*

Condition (1.4) *will be needed in the proof of Theorem 1.7. It implies in particular that* $\mathcal{H}^{d-1}(\Gamma) > 0$*. Since by assumption on the sets* K_i *we have* $QU_i \neq I$ *for every* $Q \in SO(d)$ *and every* $i \neq 1$*, it turns out that* $\dim(\ker(QU_i - I)) \leq d-1$*; therefore* (1.4) *is always satisfied when* Γ *is not contained in a hyperplane up to* $\mathcal{H}^{d-1}$*-negligible sets. If* $\dim(\ker(QU_i - I)) \leq d-2$ *for every* $Q \in SO(d)$ *and every* $i \neq 1$*, i.e., there are no rank-one connections between* K_i *and* K_1*, then* (1.4) *is satisfied whenever* $\mathcal{H}^{d-1}(\Gamma) > 0$*. Finally, note that for* $d = 1$ *the set* Ω *must be an interval and* (1.4) *implies that* $\Gamma = \partial\Omega$*. See Remark 2.9 for a discussion on the necessity of assumption* (1.4)*.*

We express the functional E_ε in terms of the displacement u by introducing the sequence of functionals $\mathcal{E}_\varepsilon \colon \mathcal{W}^{p,q}_g(\Omega;\mathbb{R}^d) \to [0,+\infty]$ defined by

$$\mathcal{E}_\varepsilon(u) := E_\varepsilon(x + \varepsilon u)\,.$$

Note that $\mathcal{E}_\varepsilon(u)$ can be written as

$$\mathcal{E}_\varepsilon(u) = \int_\Omega \Big(W(x, I + \varepsilon\nabla u) + \eta^p(\varepsilon)\,\varepsilon^p |\nabla^2 u|^p \Big)\,\mathrm{d}x\,.$$

Remark 1.2. *More in general we can consider functionals* $\hat{\mathcal{E}}_\varepsilon \colon \mathcal{W}^{p,q}_g(\Omega;\mathbb{R}^d) \to [0,+\infty]$ *of the form*

$$\hat{\mathcal{E}}_\varepsilon(u) := \int_\Omega W(x, I + \varepsilon\nabla u)\,\mathrm{d}x + \eta^p(\varepsilon)\,\varepsilon^p \mathcal{R}_p(u)\,,$$

where the regularising term $\mathcal{R}_p \colon W^{1,r}(\Omega;\mathbb{R}^d) \to [0,+\infty]$ *is such that*

$$C\int_\Omega |\nabla^2 u|^p\,\mathrm{d}x \leq \mathcal{R}_p(u) < +\infty \quad \textit{for every } u \in \mathcal{W}^{p,q}_g(\Omega;\mathbb{R}^d)\,. \tag{1.6}$$

For example, when Ω *is* C^2*,* $\Gamma = \partial\Omega$*, and* $g = 0$*, one may take* $p = 2$ *and* $\mathcal{R}_2(u) = \|\Delta u\|^2_{L^2(\Omega)}$*, or more in general* $\mathcal{R}_2(u) = \|Au\|^2_{L^2(\Omega)}$ *with* A *a uniformly elliptic operator with smooth coefficients; then the classical theory of elliptic regularity guarantees that* (1.6) *is satisfied. All the results that follow hold for* $\hat{\mathcal{E}}_\varepsilon$ *as well.*

We are interested in the case when the functional $E_\varepsilon(v)$ is complemented by an external load of the form $\varepsilon\mathcal{L}$, where $\mathcal{L}$ will be chosen in a subset of $(W^{2,p}(\Omega;\mathbb{R}^d))^*$ to be specified later on. The equilibrium configurations are then described by the minimisers of

$$E_\varepsilon(v) - \mathcal{L}(v)\,.$$

In terms of the displacement u the minimum problem reads as

$$\min_u \{\mathcal{E}_\varepsilon(u) - \varepsilon^2\mathcal{L}(u)\}\,, \tag{1.7}$$

where we dropped the term $\varepsilon\mathcal{L}(x)$, since it does not depend on u. In order to describe the asymptotic behaviour of the minimisers of (1.7), we need to introduce a set of assumptions on the scaling of $\eta(\varepsilon)$. These conditions will ensure, first, that the perturbation vanishes in the limit, second, that the minimisers are compact in some Sobolev space $W^{1,r}(\Omega;\mathbb{R}^d)$, where r

depends on the exponents p, q introduced above. Precisely, if $d = 1$ we set $r := 2$; if $d > 1$ we fix r such that

$$1 < r < p^* \qquad \text{if } p \le 2_* \text{ and } q < p^* , \tag{1.8a}$$

$$1 < r \le q \qquad \text{if } p \le 2_* \text{ and } p^* \le q < \frac{(3p-2)1^*}{p} , \tag{1.8b}$$

$$1 < r < \frac{(3p-2)1^*}{p} \qquad \text{if } p \le 2_* \text{ and } q \ge \frac{(3p-2)1^*}{p} , \tag{1.8c}$$

$$1 < r \le 2 \qquad \text{if } p > 2_* . \tag{1.8d}$$

We assume that $\eta(\varepsilon)$ satisfies

$$\lim_{\varepsilon \to 0} \eta(\varepsilon)\, \varepsilon^{1-\frac{2}{p}} = 0 , \tag{1.9a}$$

$$\lim_{\varepsilon \to 0} \frac{\eta(\varepsilon)}{\varepsilon^2} = +\infty \qquad \text{if } d = 1 , \tag{1.9b}$$

$$\eta(\varepsilon) \ge C\varepsilon^{2-\frac{r}{1^*}} \qquad \text{if } d > 1 \text{ and } r \le 1^* \vee q , \tag{1.9c}$$

$$\eta(\varepsilon) \ge C\varepsilon^{\frac{2}{r_*}-1} \qquad \text{if } d > 1 \text{ and } r > 1^* \vee q . \tag{1.9d}$$

The necessity of such assumptions is discussed in Remark 2.10 and in Section 3.

Remark 1.3. *Note that* (1.9c) *and* (1.9d) *give the same lower bound on* $\eta(\varepsilon)$ *if* $r = 1^*$ *and* $q \le 1^*$. *In contrast, if* $1^* < q < 2$ *and* $1^* < r \le q$, (1.9c) *gives a less restrictive lower bound than* (1.9d). *Moreover,* (1.9b), (1.9c), *and* (1.9d) *are compatible with* (1.9a). *More precisely, since* $p > 1$,

$$\varepsilon^{2-\frac{r}{1^*}} \le \varepsilon \ll \varepsilon^{\frac{2}{p}-1} \quad \textit{if } r \le 1^* ,$$

where the first inequality holds as equality only if $r = 1^*$. *Moreover,*

$$\varepsilon^{2-\frac{r}{1^*}} \le \varepsilon^{\frac{2}{r_*}-1} \ll \varepsilon^{\frac{2}{p}-1} \quad \textit{if } 1^* < r \le 2 \textit{ and } r < p^* ,$$

where the first inequality holds as equality only if $r = 2$, *in which case the exponent is* $2/d$. *Finally,*

$$\varepsilon^{2-\frac{r}{1^*}} \ll \varepsilon^{\frac{2}{p}-1} \quad \textit{if } r < \frac{(3p-2)1^*}{p} .$$

Note that $p^* < \frac{(3p-2)1^*}{p} < 2$ *whenever* $p < 2_*$.

After noticing that $\mathcal{W}^{p,q}_g(\Omega;\mathbb{R}^d) \subset W^{1,r}(\Omega;\mathbb{R}^d)$, we introduce the sequence of rescaled functionals $\mathcal{F}_\varepsilon \colon W^{1,r}(\Omega;\mathbb{R}^d) \to [0,+\infty]$ defined by

$$\mathcal{F}_\varepsilon(u) := \begin{cases} \dfrac{1}{\varepsilon^2}\, \mathcal{E}_\varepsilon(u) & \text{if } u \in \mathcal{W}^{p,q}_g(\Omega;\mathbb{R}^d) , \\ +\infty & \text{otherwise.} \end{cases} \tag{1.10}$$

It will be convenient to introduce also the sequence of functionals

$$F_\varepsilon(v) := \frac{1}{\varepsilon^2} E_\varepsilon(v) . \tag{1.11}$$

The main results of this paper are the following two theorems, concerning the cases of zero external load and of a general external load, respectively. As usual, $e(u)$ denotes the symmetric part of the matrix ∇u.

Theorem 1.4 (Zero external load). *Let $p > 1$, $q \in [0, 2]$, r be as in* (1.8), *and $\eta(\varepsilon)$ satisfy the scaling properties* (1.9a)–(1.9d). *Assume that W satisfies conditions* (i)–(iv) *and that Γ satisfies* (1.3) *and* (1.4). *Let $g \in W^{1,\infty}(\Omega;\mathbb{R}^d) \cap W^{2,p}(\Omega;\mathbb{R}^d)$, $\mathcal{F}_\varepsilon$ be as in* (1.10), *and*

$$m_\varepsilon := \min\{\mathcal{F}_\varepsilon(u) \colon u \in \mathcal{W}^{p,q}_g(\Omega;\mathbb{R}^d)\}\,.$$

Let $\{u_\varepsilon\}$ be a sequence such that

$$\mathcal{F}_\varepsilon(u_\varepsilon) = m_\varepsilon + o(1)\,.$$

Then $\{u_\varepsilon\}$ converges strongly in $W^{1,r}(\Omega;\mathbb{R}^d)$ to the unique solution of

$$\min\left\{\frac{1}{2}\int_\Omega D^2W(x,I)[e(u)]^2\,\mathrm{d}x \colon u \in H^1_{g,\Gamma}(\Omega;\mathbb{R}^d)\right\} =: m\,.$$

Moreover, $m_\varepsilon \to m$.

Theorem 1.5 (General external load). *Under the hypotheses of Theorem 1.4, assume in addition that $q > 1$ and*

$$\lim_{\varepsilon\to 0}\frac{\eta(\varepsilon)}{\varepsilon} = +\infty \quad \textit{if} \quad d = 1 \quad \textit{or} \quad r \le 1^*\,.$$

Let $\mathcal{L} \in (W^{1,r\wedge q}(\Omega;\mathbb{R}^d))^*$ and let

$$m^{\mathcal{L}}_\varepsilon := \min\{\mathcal{F}_\varepsilon(u) - \mathcal{L}(u) \colon u \in \mathcal{W}^{p,q}_g(\Omega;\mathbb{R}^d)\}\,.$$

Let $\{u_\varepsilon\}$ be a sequence such that

$$\mathcal{F}_\varepsilon(u_\varepsilon) = m^{\mathcal{L}}_\varepsilon + o(1)\,.$$

Then $\{u_\varepsilon\}$ converges strongly in $W^{1,r}(\Omega;\mathbb{R}^d)$ to the unique solution of

$$\min\left\{\frac{1}{2}\int_\Omega D^2W(x,I)[e(u)]^2\,\mathrm{d}x - \mathcal{L}(u) \colon u \in H^1_{g,\Gamma}(\Omega;\mathbb{R}^d)\right\} =: m^{\mathcal{L}}\,.$$

Moreover, $m^{\mathcal{L}}_\varepsilon \to m^{\mathcal{L}}$.

Theorems 1.4 and 1.5 are straightforward consequences of the next four results concerning the compactness and the Γ-convergence of the functionals $\mathcal{F}_\varepsilon$. (We refer to [8] for the definition and the main properties of Γ-convergence.) The analysis of the one-dimensional case is simpler and provides stronger compactness properties than those obtained in the case of general dimension.

Theorem 1.6 (Compactness for $d = 1$). *Let $p > 1$ and $q \in [0, 2]$. Let $\Omega = (-L, L)$, $L > 0$, and $K_i := \{k_i\}$, where $k_i \in \mathbb{R}$. Assume that W satisfies condition* (iv) *and that $\eta(\varepsilon)$ satisfies* (1.9b). *Let $g \in W^{2,p}(-L, L)$, $\mathcal{F}_\varepsilon$ and F_ε be as in* (1.10) *and* (1.11), *respectively, and let $\mathcal{L} \in (W^{1,q}(-L, L))^*$. Then*

(i) *If $v_\varepsilon \in W^{2,p}(-L, L)$ is a sequence such that $F_\varepsilon(v_\varepsilon)$ is uniformly bounded, then for every ε sufficiently small there exists $i_\varepsilon \in \{1, \dots, l\}$ such that*

$$v'_\varepsilon(x) - k_{i_\varepsilon} \to 0 \textit{ uniformly in } (-L, L)\,, \tag{1.12}$$

$$\int_{-L}^{L} |v'_\varepsilon - k_{i_\varepsilon}|^2\,\mathrm{d}x \le C\varepsilon^2 F_\varepsilon(v_\varepsilon)\,. \tag{1.13}$$

If in addition $v_\varepsilon(\pm L) = \pm k_1 L$, then $v'_\varepsilon \to k_1$ uniformly in $(-L, L)$ and

$$\int_{-L}^{L} |u'_\varepsilon|^2\,\mathrm{d}x \le C\mathcal{F}_\varepsilon(u_\varepsilon)\,, \tag{1.14}$$

where

$$u_\varepsilon(x) := \frac{v_\varepsilon(x) - k_1 x}{\varepsilon}\,. \tag{1.15}$$

In particular

$$\|u_\varepsilon\|_{H^1} \le C\,, \tag{1.16}$$

for some positive constant C independent of ε.

(ii) *Assume in addition that*

$$\lim_{\varepsilon\to 0} \frac{\eta(\varepsilon)}{\varepsilon} = +\infty\,. \tag{1.17}$$

If $v_\varepsilon \in W^{2,p}(-L, L)$ is a sequence such that $v_\varepsilon(\pm L) = \pm k_1 L$ and $\mathcal{F}_\varepsilon(u_\varepsilon) - \mathcal{L}(u_\varepsilon)$ is uniformly bounded, where u_ε is defined by (1.15)*, then* (1.12)*,* (1.13)*, and* (1.14) *still hold. Moreover,* (1.16) *holds.*

Theorem 1.7 (Compactness for $d > 1$)**.** *Let $p > 1$, $q \in [0, 2]$, r be as in* (1.8)*, and $\eta(\varepsilon)$ satisfy the scaling properties* (1.9c)*–*(1.9d)*. Assume that $d > 1$, that W satisfies condition* (iv)*, and that Γ satisfies* (1.4)*. Let $g \in W^{1,\infty}(\Omega;\mathbb{R}^d) \cap W^{2,p}(\Omega;\mathbb{R}^d)$, $\mathcal{F}_\varepsilon$ be as in* (1.10)*, and let $\mathcal{L} \in (W^{1,r\wedge q}(\Omega;\mathbb{R}^d))^*$. Then*

(i) *If $\{u_\varepsilon\}$ is a sequence in $\mathcal{W}^{p,q}_g(\Omega;\mathbb{R}^d)$ such that $\mathcal{F}_\varepsilon(u_\varepsilon)$ is uniformly bounded, then there exists a positive constant $C > 0$ such that for ε sufficiently small*

$$\int_\Omega |\nabla u_\varepsilon|^r \,\mathrm{d}x \le C\left((\mathcal{F}_\varepsilon(u_\varepsilon))^\theta + (\mathcal{F}_\varepsilon(u_\varepsilon))^{\frac{r}{2}} + \varepsilon^{2-r}\mathcal{F}_\varepsilon(u_\varepsilon) + \int_\Gamma |g|^r \,\mathrm{d}\mathcal{H}^{d-1}\right), \tag{1.18}$$

where

$$\theta := \begin{cases} 1^* = \dfrac{d}{d-1} & \textit{if } r \le 1^* \vee q\,,\\[2mm] \dfrac{r}{r_*} = \dfrac{d+r}{d} & \textit{if } r > 1^* \vee q\,. \end{cases} \tag{1.19}$$

In particular,

$$\|u_\varepsilon\|_{W^{1,r}(\Omega;\mathbb{R}^d)} \le C\,, \tag{1.20}$$

for some positive constant C independent of ε.

(ii) *Assume in addition that $q > 1$ and*

$$\lim_{\varepsilon\to 0} \frac{\eta(\varepsilon)}{\varepsilon} = +\infty \quad \textit{if} \quad r \le 1^*\,. \tag{1.21}$$

If $\{u_\varepsilon\}$ is a sequence in $\mathcal{W}^{p,q}_g(\Omega;\mathbb{R}^d)$ such that $\mathcal{F}_\varepsilon(u_\varepsilon) - \mathcal{L}(u_\varepsilon)$ is uniformly bounded, then there exists a positive constant $C > 0$ such that, for ε sufficiently small, if $r > 1^*$ (1.18) *holds, while if $r \le 1^*$ there holds*

$$\int_\Omega |\nabla u_\varepsilon|^{1^*} \,\mathrm{d}x \le C\left(o(1)\,(\mathcal{F}_\varepsilon(u_\varepsilon))^{1^*} + (\mathcal{F}_\varepsilon(u_\varepsilon))^{\frac{1^*}{2}} + \varepsilon^{2-1^*}\mathcal{F}_\varepsilon(u_\varepsilon) + \int_\Gamma |g|^{1^*} \,\mathrm{d}\mathcal{H}^{d-1}\right). \tag{1.22}$$

Moreover, (1.20) *holds.*

Theorem 1.8 (Γ-convergence)**.** *Under the hypotheses of Theorem 1.4, as $\varepsilon \to 0^+$ the sequence of functionals $\{\mathcal{F}_\varepsilon\}$ Γ-converges, with respect to the weak topology of $W^{1,r}(\Omega;\mathbb{R}^d)$, to the functional*

$$\mathcal{F}(u) := \begin{cases} \dfrac{1}{2}\displaystyle\int_\Omega D^2W(x,I)[e(u)]^2\,\mathrm{d}x & \text{if } u \in H^1_{g,\Gamma}(\Omega;\mathbb{R}^d)\,, \\ +\infty & \text{otherwise.} \end{cases} \tag{1.23}$$

Theorem 1.9 (Strong convergence of recovery sequences)**.** *Under the hypotheses of Theorem 1.8, let $\varepsilon_j \to 0$ and let $\{u_j\}$ be a recovery sequence for $u \in H^1_{g,\Gamma}(\Omega;\mathbb{R}^d)$, that is, $u_j \rightharpoonup u$ weakly in $W^{1,r}(\Omega;\mathbb{R}^d)$ and $\mathcal{F}_{\varepsilon_j}(u_j) \to \mathcal{F}(u)$. Then $u_j \to u$ strongly in $W^{1,r}(\Omega;\mathbb{R}^d)$.*

Table 1. Choice of the exponent r

	$0 \le q < p^*$	$p^* \le q < \frac{(3p-2)1^*}{p}$	$\frac{(3p-2)1^*}{p} \le q \le 2$
$1 < p \le 2_*$	$1 < r < p^*$	$1 < r \le q$	$1 < r < \frac{(3p-2)1^*}{p} \le 2$
$2_* < p < +\infty$	$1 < r \le 2$		

Table 2. Range for η ensuring compactness and its optimality

	zero load ($0 \le q \le 2$)	nonzero load ($1 < q \le 2$)	optimality
$1 < r \le 1^*$	$C\varepsilon^{2-\frac{r}{1^*}} \le \eta(\varepsilon) \ll \varepsilon^{\frac{2}{p}-1}$	$\varepsilon \ll \eta(\varepsilon) \ll \varepsilon^{\frac{2}{p}-1}$	yes
$1^* < r \le q$	$C\varepsilon^{2-\frac{r}{1^*}} \le \eta(\varepsilon) \ll \varepsilon^{\frac{2}{p}-1}$		yes
$1^* \vee q < r < 2$	$C\varepsilon^{2-\frac{r}{1^*}} \ll C\varepsilon^{\frac{2}{r_*}-1} \le \eta(\varepsilon) \ll \varepsilon^{\frac{2}{p}-1}$		unknown
$r = 2$	$C\varepsilon^{\frac{2}{d}} \le \eta(\varepsilon) \ll \varepsilon^{\frac{2}{p}-1}$		yes

2. Proof of the main results

2.1. **Compactness in the one-dimensional case.** We first treat the problem in dimension one. Although the analysis is simpler and the results are stronger than in larger dimension, it already contains some of the essential features of the analysis in the general case. In this section we prove the first part of Theorem 1.6 which concerns the case of zero external load. The case of general external load will be proven in Section 2.4.

Proof of Theorem 1.6 (i). Fix $\delta > 0$ such that $\delta < \frac{1}{4}|k_i - k_j|$ for every $i \neq j = 1, \dots, l$, and set $\Omega_\varepsilon := \{\mathrm{dist}(v'_\varepsilon, K) < \delta\}$. By hypothesis (iv) one has

$$|\Omega \backslash \Omega_\varepsilon| \leq C\varepsilon^2 F_\varepsilon(v_\varepsilon)\,.$$

Therefore for ε sufficiently small one can find $i_\varepsilon \in \{1, \dots, l\}$ and $x^1_\varepsilon \in (-L, L)$ such that $|v'_\varepsilon(x^1_\varepsilon) - k_{i_\varepsilon}| < \delta$. We claim that $|v'_\varepsilon(x) - k_{i_\varepsilon}| < 2\delta$ for all $x \in (-L, L)$. Indeed, suppose on the contrary that there exists a sequence of points $x^2_\varepsilon \in (-L, L)$ such that $|v'_\varepsilon(x^2_\varepsilon) - k_{i_\varepsilon}| > 2\delta$ and set

$$\widetilde{W}(z) := f_q(\mathrm{dist}(z, K)), \quad z \in \mathbb{R}^d,$$

where f_q is defined in hypothesis (iv) on W. Then, Young's inequality in combination with $v'_\varepsilon \in C((-L, L))$ would yield

$$\begin{aligned} F_\varepsilon(v_\varepsilon) &\geq \frac{\eta}{\varepsilon^2}\int_{-L}^{L} \widetilde{W}^{\frac{p-1}{p}}(v'_\varepsilon(x))|v''_\varepsilon(x)|\,\mathrm{d}x \geq \frac{\eta}{\varepsilon^2}\Big|\int_{x^1_\varepsilon}^{x^2_\varepsilon} \widetilde{W}^{\frac{p-1}{p}}(v'_\varepsilon(x))\,v''_\varepsilon(x)\,\mathrm{d}x\Big| \\ &\geq \frac{\eta}{\varepsilon^2}\Big|\int_{v'_\varepsilon(x^1_\varepsilon)}^{v'_\varepsilon(x^2_\varepsilon)} \widetilde{W}^{\frac{p-1}{p}}(z)\,\mathrm{d}z\Big| \geq C\frac{\eta}{\varepsilon^2}\,, \end{aligned} \tag{2.1}$$

which contradicts the uniform bound on $F_\varepsilon(v_\varepsilon)$. Remark that in the last inequality of (2.1) we used the fact that, by the growth condition (iv),

$$\left|\int_{v'_\varepsilon(x^1_\varepsilon)}^{v'_\varepsilon(x^2_\varepsilon)} \widetilde{W}^{\frac{p-1}{p}}(z)\,\mathrm{d}z\right| > C > 0\,,$$

where the constant C depends only on δ. By the arbitrariness of δ we then deduce (1.12). Using (1.12) and the quadratic growth assumption of W near the wells we obtain, for δ and ε sufficiently small,

$$\int_{-L}^{L} |v'_\varepsilon - k_{i_\varepsilon}|^2\,\mathrm{d}x \leq C\int_{-L}^{L} W(x, v'_\varepsilon)\,\mathrm{d}x \leq C\varepsilon^2 F_\varepsilon(v_\varepsilon)\,,$$

thus (1.13) holds. Assume now the boundary condition $v_\varepsilon(\pm L) = \pm k_1 L$. Then (1.12) holds with $i_\varepsilon = 1$ and (1.14) follows from (1.13). □

2.2. **One-well lower bound.** A fundamental step in the proof of Theorem 1.7 is the result proven in this section, stated in Theorem 2.2, which allows us to identify a single energy well where the deformation gradient lies in most of the domain. We underline that the result of Theorem 2.2 holds true in a slightly more general situation than that presented above. In fact, assumptions (i)–(iii), (1.3), and (1.4) are not needed; the deformations may be of the form $u\colon \Omega \to \mathbb{R}^m$ with m possibly different from d; the energy wells may be compact sets of matrices, without requirements on their structure.

For the reader's convenience we recall the following variant of the Poincaré inequality, which follows e.g. from [18, Theorem 4.4.2].

Lemma 2.1. *Let $s \geq 1$. For every constant $c > 0$ there exists a constant $K = K(c) > 0$ such that*

$$\|f\|_{L^s(\Omega)} \leq K\|\nabla f\|_{L^s(\Omega;\mathbb{R}^{d\times d})}$$

for every $f \in H^1(\Omega)$ such that $|\{f = 0\}| \geq c$.

In the next theorem, for every measurable set $A \subset \Omega$ we denote by $F_\varepsilon(\cdot, A)$ the functionals defined as in (1.11) by replacing Ω with A only in the first integral.

Theorem 2.2 (One-well lower bound)**.** *Let $p > 1$, $q \in [0, 2]$, r be as in* (1.8)*, and $\eta(\varepsilon)$ satisfy the scaling properties* (1.9c)–(1.9d)*. Assume that $d > 1$ and that W satisfies condition* (iv)*. Let F_ε be as in* (1.11)*. Let $\{v_\varepsilon\}$ be a sequence in $W^{2,p}(\Omega;\mathbb{R}^d)$ such that $\lim_{\varepsilon\to 0}\varepsilon^2 F_\varepsilon(v_\varepsilon) = 0$. Then there exist $C > 0$ and $\delta > 0$ such that for every $\varepsilon > 0$ sufficiently small there is $i_\varepsilon \in \{1, \dots, l\}$ with*

$$\frac{1}{\varepsilon^r}\int_A \mathrm{dist}^r(\nabla v_\varepsilon, K_{i_\varepsilon})\,\mathrm{d}x \le C(F_\varepsilon(v_\varepsilon, A))^{\frac{r}{2}} \quad \textit{for every measurable set } A \subset \Omega_\varepsilon^{i_\varepsilon}\,, \tag{2.2a}$$

$$\frac{1}{\varepsilon^r}\int_{\Omega\backslash\Omega_\varepsilon^{i_\varepsilon}} \mathrm{dist}^r(\nabla v_\varepsilon, K_{i_\varepsilon})\,\mathrm{d}x \le C\Big((F_\varepsilon(v_\varepsilon, \Omega\backslash\Omega_\varepsilon^{i_\varepsilon}))^\theta + \varepsilon^{2-r}F_\varepsilon(v_\varepsilon, \Omega\backslash\Omega_\varepsilon^{i_\varepsilon})\Big)\,, \tag{2.2b}$$

where $\Omega_\varepsilon^i := \{\mathrm{dist}(\nabla v_\varepsilon, K_i) \le \delta\}$ for $i = 1, \dots, l$ and θ is defined in (1.19)*.*

Remark 2.3. *Note that, in the case of two strongly incompatible wells, the rigidity estimate by Chaudhuri and Müller* [6] *provides the one-well lower bound of Theorem 2.2 without the regularising term, i.e., one may take in this case $\eta = 0$ in* (1.11)*.*

Proof of Theorem 2.2. We split the proof into three main steps.

Step 1: Estimate away from the wells. We fix a constant $\delta < 1 \wedge \min_{i\neq j}\{\frac{1}{2}\mathrm{dist}(K_i, K_j)\}$. In this proof the constants denoted by C, whose value changes from place to place, may depend on δ but not on ε.

Set

$$\Omega_\varepsilon := \{\mathrm{dist}(\nabla v_\varepsilon, K) \le \delta\}\,. \tag{2.3}$$

By hypothesis (iv) we get that $W(x, \nabla v_\varepsilon) \ge f_q(\delta)$ a.e. on $\Omega\backslash\Omega_\varepsilon$ and in particular

$$|\Omega\backslash\Omega_\varepsilon| < C\varepsilon^2 F_\varepsilon(v_\varepsilon, \Omega\backslash\Omega_\varepsilon) \tag{2.4}$$

for some positive constant $C = C(\delta)$ independent of ε.

We observe that, for any $i \in \{1, \dots, l\}$,

$$\mathrm{dist}(\nabla v_\varepsilon, K_i) \le \mathrm{dist}(\nabla v_\varepsilon, K) + \mathrm{diam}\,K \le \left(1 + \frac{\mathrm{diam}\,K}{\delta}\right)\mathrm{dist}(\nabla v_\varepsilon, K) \quad \text{in } \Omega\backslash\Omega_\varepsilon\,.$$

Hence, using also the growth assumption (iv) on W, we obtain for some $C = C(\delta)$

$$\max_{i\in\{1,\dots l\}}\int_{\Omega\backslash\Omega_\varepsilon} \mathrm{dist}^q(\nabla v_\varepsilon, K_i)\,\mathrm{d}x \le C\int_{\Omega\backslash\Omega_\varepsilon} \mathrm{dist}^q(\nabla v_\varepsilon, K)\,\mathrm{d}x \le C\varepsilon^2 F_\varepsilon(v_\varepsilon)\,.$$

Step 2: Estimate near the majority phase. Note that $\Omega_\varepsilon = \bigcup_{i=1}^l \Omega_\varepsilon^i$. By (2.4), there exists $i_\varepsilon \in \{1, \dots, l\}$ such that for ε sufficiently small

$$|\Omega_\varepsilon^{i_\varepsilon}| \ge C$$

for some positive constant $C = C(\delta)$ independent of ε. The quadratic growth assumption of W near the wells in combination with Hölder's inequality yields

$$\int_A \mathrm{dist}^r(\nabla v_\varepsilon, K_{i_\varepsilon})\,\mathrm{d}x \le C\left(\int_A \mathrm{dist}^2(\nabla v_\varepsilon, K_{i_\varepsilon})\,\mathrm{d}x\right)^{\frac{r}{2}} \le C\varepsilon^r(F_\varepsilon(v_\varepsilon, A))^{\frac{r}{2}}\,,$$

for every measurable set $A \subset \Omega_\varepsilon^{i_\varepsilon}$, thus (2.2a) holds.

Step 3: Estimate away from the majority phase. Set, for $F \in \mathbb{R}^{d\times d}$,

$$\widetilde{W}(F) := f_q(\mathrm{dist}(F, K))\,,$$

where f_q is defined in hypothesis (iv) on W. Observe that, by Young's inequality and hypothesis (iv) on W, we have

$$\int_\Omega \big(\widetilde{W}(\nabla v_\varepsilon)\big)^{\frac{1}{\alpha}} |\nabla^2 v_\varepsilon|^{\frac{p}{\beta}}\,\mathrm{d}x \le C\frac{1}{\eta^{\frac{p}{\beta}}}\int_\Omega \big(W(x,\nabla v_\varepsilon)\big)^{\frac{1}{\alpha}}\big(\eta^p|\nabla^2 v_\varepsilon|^p\big)^{\frac{1}{\beta}}\,\mathrm{d}x \le C\frac{\varepsilon^2}{\eta^{\frac{p}{\beta}}}F_\varepsilon(v_\varepsilon) \tag{2.5}$$

for each $\alpha,\beta>1$ such that $1/\alpha+1/\beta=1$ and for a constant $C=C(\beta)$. Fix such α and β and define the distance

$$d_{\widetilde{W}}(F,G) := \inf\bigg\{\int_0^1 (\widetilde{W}(\xi(s)))^{\frac{\beta}{\alpha p}}|\xi'(s)|\,\mathrm{d}s \,:\, \xi\in C^1([0,1];\mathbb{R}^{d\times d}),\ \xi(0)=F\,,\ \xi(1)=G\bigg\}. \tag{2.6}$$

It can be easily checked that

$$d_{\widetilde{W}}(\nabla v_\varepsilon, K_{i_\varepsilon}) < \tilde\delta := \delta^{2\frac{\beta}{\alpha p}+1} \quad \text{in } \Omega_\varepsilon^{i_\varepsilon}\,. \tag{2.7}$$

Let

$$\tilde h_\varepsilon(x) := (d_{\widetilde{W}}(\nabla v_\varepsilon(x), K_{i_\varepsilon}) - \tilde\delta)\vee 0\,.$$

Note that $\tilde h_\varepsilon\equiv 0$ on $\Omega_\varepsilon^{i_\varepsilon}$ and that $|\nabla\tilde h_\varepsilon| \le (\widetilde{W}(\nabla v_\varepsilon))^{\frac{\beta}{\alpha p}}|\nabla^2 v_\varepsilon|$. Therefore, assuming $\beta\le p$ and setting $\gamma:=p/\beta$, by (2.5) we have $\tilde h_\varepsilon\in W^{1,\gamma}(\Omega;\mathbb{R}^d)$ and

$$\int_\Omega |\nabla\tilde h_\varepsilon|^\gamma\,\mathrm{d}x \le \int_{\Omega\setminus\Omega_\varepsilon^{i_\varepsilon}} (\widetilde{W}(\nabla v_\varepsilon))^{\frac{1}{\alpha}}|\nabla^2 v_\varepsilon|^\gamma\,\mathrm{d}x \le C\frac{\varepsilon^2}{\eta^\gamma}F_\varepsilon(v_\varepsilon,\Omega\setminus\Omega_\varepsilon^{i_\varepsilon})\,.$$

Choose now β in such a way that $1\le\gamma<d$. By the Poincaré inequality (Lemma 2.1) and the Sobolev immersion, we have

$$\int_\Omega \tilde h_\varepsilon^{\gamma^*}\,\mathrm{d}x \le C\left(\int_\Omega |\nabla\tilde h_\varepsilon|^\gamma\,\mathrm{d}x\right)^{\frac{\gamma^*}{\gamma}} \le C\frac{\varepsilon^{2\frac{\gamma^*}{\gamma}}}{\eta^{\gamma^*}}(F_\varepsilon(v_\varepsilon),\Omega\setminus\Omega_\varepsilon^{i_\varepsilon})^{\frac{\gamma^*}{\gamma}}\,. \tag{2.8}$$

Next set $\widetilde\Omega_\varepsilon^{i_\varepsilon} := \{d_{\widetilde{W}}(\nabla v_\varepsilon(x),K_{i_\varepsilon})\le 2\tilde\delta\}$ and note that (2.7) implies $\Omega_\varepsilon^{i_\varepsilon}\subset\widetilde\Omega_\varepsilon^{i_\varepsilon}$. By Lemma 2.4 one has that $\mathrm{dist}(\nabla v_\varepsilon(x),K_{i_\varepsilon})\le C\,d_{\widetilde{W}}(\nabla v_\varepsilon(x),K_{i_\varepsilon})\le C\tilde h_\varepsilon(x)$ in $\Omega\setminus\widetilde\Omega_\varepsilon^{i_\varepsilon}$. Moreover, Lemma 2.5 shows that $\mathrm{dist}(\nabla v_\varepsilon(x),K_{i_\varepsilon})$ is equi-bounded in $\widetilde\Omega_\varepsilon^{i_\varepsilon}\setminus\Omega_\varepsilon^{i_\varepsilon}$. Let us further refine the choice of δ in such a way that $\widetilde\Omega_\varepsilon^{i_\varepsilon}\setminus\Omega_\varepsilon^{i_\varepsilon}\subset\Omega\setminus\Omega_\varepsilon\subset\Omega\setminus\Omega_\varepsilon^{i_\varepsilon}$. Hence, by (2.8) and (2.4), we get

$$\begin{aligned}\int_{\Omega\setminus\Omega_\varepsilon^{i_\varepsilon}} \mathrm{dist}^{\gamma^*}(\nabla v_\varepsilon,K_{i_\varepsilon})\,\mathrm{d}x &= \int_{\Omega\setminus\widetilde\Omega_\varepsilon^{i_\varepsilon}} \mathrm{dist}^{\gamma^*}(\nabla v_\varepsilon,K_{i_\varepsilon})\,\mathrm{d}x + \int_{\widetilde\Omega_\varepsilon^{i_\varepsilon}\setminus\Omega_\varepsilon^{i_\varepsilon}} \mathrm{dist}^{\gamma^*}(\nabla v_\varepsilon,K_{i_\varepsilon})\,\mathrm{d}x\\ &\le C\left(\int_\Omega \tilde h_\varepsilon^{\gamma^*}\,\mathrm{d}x + |\Omega\setminus\Omega_\varepsilon|\right) \le C\left(\frac{\varepsilon^{2\frac{\gamma^*}{\gamma}}}{\eta^{\gamma^*}}(F_\varepsilon(v_\varepsilon),\Omega\setminus\Omega_\varepsilon^{i_\varepsilon})^{\frac{\gamma^*}{\gamma}} + \varepsilon^2F_\varepsilon(v_\varepsilon,\Omega\setminus\Omega_\varepsilon^{i_\varepsilon})\right)\,.\end{aligned} \tag{2.9}$$

If $r\le 1^*$, then we choose $\beta=p>1$, so $\gamma=1$. By (1.9c) and (2.9), using the fact that $\mathrm{dist}(\nabla v_\varepsilon,K_{i_\varepsilon})\ge\delta$ in $\Omega\setminus\Omega_\varepsilon^{i_\varepsilon}$, we obtain

$$\begin{aligned}\int_{\Omega\setminus\Omega_\varepsilon^{i_\varepsilon}} \mathrm{dist}^r(\nabla v_\varepsilon,K_{i_\varepsilon})\,\mathrm{d}x &\le C\int_{\Omega\setminus\Omega_\varepsilon^{i_\varepsilon}} \mathrm{dist}^{1^*}(\nabla v_\varepsilon,K_{i_\varepsilon})\,\mathrm{d}x\\ &\le C\left(\varepsilon^r(F_\varepsilon(v_\varepsilon),\Omega\setminus\Omega_\varepsilon^{i_\varepsilon})^{1^*} + \varepsilon^2F_\varepsilon(v_\varepsilon,\Omega\setminus\Omega_\varepsilon^{i_\varepsilon})\right).\end{aligned}$$

If $1^* < r \le q$, fix $M > 0$ such that $|F| < M$ for every $F \in K$ and set $B_\varepsilon^M := \{|\nabla v_\varepsilon| \le M\}$. Then, using the growth condition (iv) on W and using (1.9c) and (2.9) with $\gamma = 1$ as above, we deduce

$$\begin{aligned}\int_{\Omega\setminus\Omega_\varepsilon^{i_\varepsilon}} \operatorname{dist}^r(\nabla v_\varepsilon, K_{i_\varepsilon})\,\mathrm{d}x &= \int_{(\Omega\setminus\Omega_\varepsilon^{i_\varepsilon})\cap B_\varepsilon^M} \operatorname{dist}^r(\nabla v_\varepsilon, K_{i_\varepsilon})\,\mathrm{d}x + \int_{(\Omega\setminus\Omega_\varepsilon^{i_\varepsilon})\setminus B_\varepsilon^M} \operatorname{dist}^r(\nabla v_\varepsilon, K_{i_\varepsilon})\,\mathrm{d}x \\ &\le C\int_{(\Omega\setminus\Omega_\varepsilon^{i_\varepsilon})\cap B_\varepsilon^M} \operatorname{dist}^{1^*}(\nabla v_\varepsilon, K_{i_\varepsilon})\,\mathrm{d}x + C\int_{(\Omega\setminus\Omega_\varepsilon^{i_\varepsilon})\setminus B_\varepsilon^M} \operatorname{dist}^q(\nabla v_\varepsilon, K_{i_\varepsilon})\,\mathrm{d}x \\ &\le C\int_{(\Omega\setminus\Omega_\varepsilon^{i_\varepsilon})\cap B_\varepsilon^M} \operatorname{dist}^{1^*}(\nabla v_\varepsilon, K_{i_\varepsilon})\,\mathrm{d}x + C\int_{(\Omega\setminus\Omega_\varepsilon^{i_\varepsilon})\setminus B_\varepsilon^M} W(x,\nabla v_\varepsilon)\,\mathrm{d}x \\ &\le C\left(\varepsilon^r (F_\varepsilon(v_\varepsilon), \Omega\setminus\Omega_\varepsilon^{i_\varepsilon})^{1^*} + \varepsilon^2 F_\varepsilon(v_\varepsilon, \Omega\setminus\Omega_\varepsilon^{i_\varepsilon})\right).\end{aligned}$$

If $r > 1^* \vee q$, then (1.8) implies $r_* < p$. Choosing $\beta = p/r_* > 1$ we have $\gamma = r_*$ and $\gamma^* = r$. Therefore, (1.9d) and (2.9) give

$$\int_{\Omega\setminus\Omega_\varepsilon^{i_\varepsilon}} \operatorname{dist}^r(\nabla v_\varepsilon, K_{i_\varepsilon})\,\mathrm{d}x \le C\left(\varepsilon^r (F_\varepsilon(v_\varepsilon), \Omega\setminus\Omega_\varepsilon^{i_\varepsilon})^{\frac{r}{r_*}} + \varepsilon^2 F_\varepsilon(v_\varepsilon, \Omega\setminus\Omega_\varepsilon^{i_\varepsilon})\right).$$

Thus (2.2b) is proven. □

In the next lemmas $d_{\widetilde{W}}$ denotes the distance function defined in (2.6).

Lemma 2.4. *Let $\tilde\delta > 0$ and $i \in \{1,\dots,l\}$. There exists $C = C(\tilde\delta)$ such that*

$$\operatorname{dist}(F, K_i) \le C(\tilde\delta)\, d_{\widetilde{W}}(F, K_i) \quad \textit{for every } F \in \mathbb{R}^{d\times d} \textit{ such that } d_{\widetilde{W}}(F, K_i) \ge \tilde\delta\,.$$

Proof. Let $R > 0$ be sufficiently large to ensure $K \subset B_R(0) \subset \mathbb{R}^{d\times d}$. If $|F| \le 2R$, then

$$\operatorname{dist}(F, K_i) \le 3R = \frac{3R}{\tilde\delta}\tilde\delta \le \frac{3R}{\tilde\delta}\, d_{\widetilde{W}}(F, K_i)\,.$$

On the other hand, if $|F| > 2R$, then

$$\operatorname{dist}(F, K_i) \le |F| + R < \frac{3}{2}|F|\,.$$

Hence, to get the conclusion, it suffices to prove that

$$|F| \le C\, d_{\widetilde{W}}(F, K_i) \quad \text{if } |F| > 2R\,.$$

To this end, let $|F| > 2R$ and $G \in K_i$ be such that $d_{\widetilde{W}}(F, K_i) = d_{\widetilde{W}}(F, G)$. Fix $\tau > 0$ and let $\xi \in C^1([0,1];\mathbb{R}^{d\times d})$ be a quasi-minimiser for (2.6), namely, $\xi(0) = F$, $\xi(1) = G$, and $d_{\widetilde{W}}(F, G) > \int_0^1 (\widetilde{W}(\xi(s)))^{\frac{\beta}{\alpha d}} |\xi'(s)|\,\mathrm{d}s - \tau$. Set $t_0 := \min\{t \in [0,1] \colon |\xi(t)| \le R\}$. Note that $|\xi(t_0)| = R$. Hence, recalling assumption (iv), setting $\overline{R} := \min\{\operatorname{dist}(F', K) \colon |F'| \ge R\} = \operatorname{dist}(B_R(0)^c, K)$, and observing that $\overline{R} > 0$ since $K \subset B_R(0)$, we get

$$\begin{aligned} d_{\widetilde{W}}(F, K_i) &\ge (f_q(\overline{R}))^{\frac{\beta}{\alpha d}} \int_0^{t_0} |\xi'(s)|\,\mathrm{d}s - \tau \ge (f_q(\overline{R}))^{\frac{\beta}{\alpha d}} |F - \xi(t_0)| - \tau \\ &\ge (f_q(\overline{R}))^{\frac{\beta}{\alpha d}} (|F| - R) - \tau \ge \frac{(f_q(\overline{R}))^{\frac{\beta}{\alpha d}}}{2}|F| - \tau\,. \end{aligned}$$

The thesis follows from the arbitrariness of τ. □

Lemma 2.5. *Let $\tilde\delta > 0$ and $i \in \{1,\dots,l\}$. There exists $C = C(\tilde\delta)$ such that*

$$\operatorname{dist}(F, K_i) \le C(\tilde\delta) \quad \textit{for every } F \in \mathbb{R}^{d\times d} \textit{ such that } d_{\widetilde{W}}(F, K_i) < \tilde\delta\,.$$

Proof. Fix $\delta > 0$. We only have to consider the case $\mathrm{dist}(F, K_i) > \delta$. Let $G \in K_i$ be such that $d_{\widetilde{W}}(F, K_i) = d_{\widetilde{W}}(F, G)$. Fix $\tau > 0$ and let $\xi \in C^1([0,1]; \mathbb{R}^{d\times d})$ be a quasi-minimiser for (2.6), namely, $\xi(0) = F$, $\xi(1) = G$, and $d_{\widetilde{W}}(F, G) > \int_0^1 (\widetilde{W}(\xi(s)))^{\frac{\beta}{\alpha d}} |\xi'(s)| \,\mathrm{d}s - \tau$. Let $r := \mathrm{dist}(F, K_i)$ and $t_0 := \min\{t \in [0,1] \colon \mathrm{dist}(\xi(t), K_i) \le \frac{r}{2}\}$. Recalling that $r = \mathrm{dist}(F, K_i) > \delta$ and that $\mathrm{dist}(\xi(t_0), K_i) = \frac{r}{2}$, we obtain

$$\begin{aligned}\tilde{\delta} > d_{\widetilde{W}}(F, K_i) &\ge (f_q(\tfrac{d}{2}))^{\frac{\beta}{\alpha d}} \int_0^{t_0} |\xi'(s)| \,\mathrm{d}s \,-\, \tau \ge (f_q(\tfrac{\delta}{2}))^{\frac{\beta}{\alpha d}} |F - \xi(t_0)| \,-\, \tau \\ &\ge (f_q(\tfrac{\delta}{2}))^{\frac{\beta}{\alpha d}} \,\tfrac{d}{2} \,-\, \tau = \frac{(f_q(\frac{\delta}{2}))^{\frac{\beta}{\alpha d}}}{2} \,\mathrm{dist}(F, K_i) \,-\, \tau\,.\end{aligned}$$

The thesis follows. □

Remark 2.6. *Let v_ε and i_ε be as in Theorem 2.2. Then there exists a subsequence $\varepsilon_k \to 0$ such that $i_{\varepsilon_k} = \bar{\imath}$ for every k. Assume in addition that the right-hand sides of* (2.2a)–(2.2b) *multiplied by ε^r tend to zero (which always holds e.g. when $F_\varepsilon(v_\varepsilon)$ is uniformly bounded). Since $\mathrm{dist}(\nabla v_{\varepsilon_k}, K_{\bar{\imath}}) \ge \delta$ in $\Omega \backslash \Omega^{\bar{\imath}}_{\varepsilon_k}$, estimate* (2.2b) *implies that $|\Omega^{\bar{\imath}}_{\varepsilon_k}| \to |\Omega|$ for $\varepsilon_k \to 0$.*

Moreover, the rigidity estimate of Theorem 2.8 below, in combination with (2.2a)–(2.2b), *implies that for every ε_k there is $Q_{\varepsilon_k} \in K_{\bar{\imath}}$ such that*

$$\int_\Omega |\nabla v_{\varepsilon_k} - Q_{\varepsilon_k}|^r \,\mathrm{d}x \le C \int_\Omega \mathrm{dist}^r(\nabla v_{\varepsilon_k}, K_{\bar{\imath}}) \,\mathrm{d}x \to 0 \quad \text{as } \varepsilon_k \to 0\,,$$

for a constant C independent of ε_k. We deduce that $\nabla v_{\varepsilon_k} \to Q$ in $L^r(\Omega; \mathbb{R}^{d\times d})$, for some $Q \in K_{\bar{\imath}}$. Hence, even if the wells are compatible, in the scaling regime (1.9c)–(1.9d) *transitions between different energy wells are not allowed in the limit as $\varepsilon \to 0$. This is independent of the boundary condition.*

In the following section we will see that fixing a boundary condition as in (1.2) *and* (1.5) *determines the energy well $K_{\bar{\imath}}$.*

Remark 2.7. *In the case $p > d$, one can prove a stronger version of Theorem 2.2, analogous to the one-dimensional case, by assuming that $F_\varepsilon(v_\varepsilon)$ is uniformly bounded and that $\eta(\varepsilon)$ satisfies*

$$\lim_{\varepsilon\to 0} \frac{\eta}{\varepsilon^{\frac{2}{p}}} = +\infty\,.$$

Notice that the last condition is stronger than both (1.9c) *and* (1.9d), *since $\frac{2}{p} < \frac{2}{d} \le \frac{2}{d} + \frac{2}{r} - 1 = \frac{2}{r_*} - 1$. Under such assumptions, one can prove that the deformation gradient ∇v_ε is close to one well at each point of Ω. Namely, for ε sufficiently small we have*

$$h_\varepsilon(x) := (\mathrm{dist}(\nabla v_\varepsilon(x), K_{i_\varepsilon}) - \delta) \vee 0 \le \delta \quad \text{for every } x \in \Omega\,.$$

Indeed, if a sequence of points x_ε existed such that $h_\varepsilon(x_\varepsilon) > \delta$, then Morrey's and Poincaré's inequalities would yield the contradiction

$$\delta^p \le C \|h_\varepsilon\|^p_{C^{0,t}(\Omega)} \le C \|h_\varepsilon\|^p_{W^{1,p}(\Omega)} \le C \|\nabla h_\varepsilon\|^p_{L^p(\Omega)} \le C \frac{\varepsilon^2}{\eta^p}\,,$$

where $t = \frac{p-d}{p}$.

2.3. **Compactness in the case $d > 1$.** In this section we deal with the case of zero external load and, exploiting the boundary condition (1.5), we prove Theorem 1.7 (i). We employ the following well-known result.

Theorem 2.8. [12, Theorem 3.1] *Let $d > 1$ and $s \in (1, +\infty)$. Suppose that $U \subset \mathbb{R}^d$ is a bounded Lipschitz domain. Then there exists a constant $C = C(U)$ such that for each $u \in W^{1,s}(U; \mathbb{R}^d)$ there exists a constant matrix $R \in SO(d)$ such that*

$$\|\nabla u - R\|_{L^s(U;\mathbb{R}^{d\times d})} \le C(U) \|\mathrm{dist}(\nabla u, SO(d))\|_{L^s(U)} \,. \tag{2.10}$$

The constant $C(U)$ is invariant under dilation and translation of the domain.

Proof of Theorem 1.7 (i). From Theorem 2.2, by choosing $A = \Omega_\varepsilon^{i_\varepsilon}$, we know that for each $\varepsilon > 0$ there exists $i_\varepsilon \in \{1, \ldots, l\}$ such that

$$\frac{1}{\varepsilon^r} \int_\Omega \mathrm{dist}^r(\nabla v_\varepsilon, K_{i_\varepsilon}) \, \mathrm{d}x \le C \Big((F_\varepsilon(v_\varepsilon))^\theta + (F_\varepsilon(v_\varepsilon))^{\frac{r}{2}} + \varepsilon^{2-r} F_\varepsilon(v_\varepsilon) \Big) \,,$$

where F_ε is as in (1.11). We claim that $K_{i_\varepsilon} = SO(d)$ for ε small enough. Assuming that the claim is true, we complete the proof following [9]. Upon application of the Rigidity Estimate (2.10), we find a sequence $Q_\varepsilon \in SO(d)$ such that

$$\begin{aligned} \frac{1}{\varepsilon^r} \int_\Omega |\nabla v_\varepsilon - Q_\varepsilon|^r \, \mathrm{d}x &\le \frac{C}{\varepsilon^r} \int_\Omega \mathrm{dist}^r(\nabla v_\varepsilon, SO(d)) \, \mathrm{d}x \\ &\le C \Big((F_\varepsilon(v_\varepsilon))^\theta + (F_\varepsilon(v_\varepsilon))^{\frac{r}{2}} + \varepsilon^{2-r} F_\varepsilon(v_\varepsilon) \Big) \,. \end{aligned} \tag{2.11}$$

Set $\zeta_\varepsilon := \fint_\Omega (v_\varepsilon - Q_\varepsilon x) \, \mathrm{d}x$. Then by (2.11), the Poincaré-Wirtinger inequality, and the continuity of the trace operator, we have

$$\begin{aligned} \int_\Gamma |x - Q_\varepsilon x - \zeta_\varepsilon|^r \, \mathrm{d}\mathcal{H}^{d-1} &\le C \int_\Omega |\nabla v_\varepsilon - Q_\varepsilon|^r \, \mathrm{d}x + C\varepsilon^r \int_\Gamma |g|^r \, \mathrm{d}\mathcal{H}^{d-1} \\ &\le C\varepsilon^r \Big((F_\varepsilon(v_\varepsilon))^\theta + (F_\varepsilon(v_\varepsilon))^{\frac{r}{2}} + \varepsilon^{2-r} F_\varepsilon(v_\varepsilon) \Big) + C\varepsilon^r \int_\Gamma |g|^r \, \mathrm{d}\mathcal{H}^{d-1} \,. \end{aligned} \tag{2.12}$$

Arguing as in [9, Lemma 3.3 and Proposition 3.4], one gets

$$|I - Q_\varepsilon|^r \le C \int_\Gamma |x - Q_\varepsilon x - \zeta_\varepsilon|^r \, \mathrm{d}\mathcal{H}^{d-1} \,.$$

Thus, by (2.11) and (2.12),

$$\int_\Omega |\nabla v_\varepsilon - I|^r \, \mathrm{d}x \le C\varepsilon^r \Big((F_\varepsilon(v_\varepsilon))^\theta + (F_\varepsilon(v_\varepsilon))^{\frac{r}{2}} + \varepsilon^{2-r} F_\varepsilon(v_\varepsilon) \Big) + C\varepsilon^r \int_\Gamma |g|^r \, \mathrm{d}\mathcal{H}^{d-1} \,. \tag{2.13}$$

Replacing $\nabla v_\varepsilon = I + \varepsilon \nabla u_\varepsilon$ in the previous inequality yields (1.18). The latter implies in its turn (1.20) since $\mathcal{F}_\varepsilon(u_\varepsilon)$ is uniformly bounded.

We are left to prove the claim. Suppose on the contrary that there exists $i \in \{2, \ldots N\}$ such that $i_\varepsilon = i$ for some subsequence (not relabelled). Then we may apply the Rigidity Estimate (2.10) with $SO(d)U_i$ in place of $SO(d)$ and find a sequence $\tilde{Q}_\varepsilon \in SO(d)$ such that

$$\begin{aligned} \int_\Omega |\nabla v_\varepsilon - \tilde{Q}_\varepsilon U_i|^r \, \mathrm{d}x &\le C \int_\Omega \mathrm{dist}^r(\nabla v_\varepsilon, SO(d)U_i) \, \mathrm{d}x \\ &\le C\varepsilon^r \Big((F_\varepsilon(v_\varepsilon))^\theta + (F_\varepsilon(v_\varepsilon))^{\frac{r}{2}} + \varepsilon^{2-r} F_\varepsilon(v_\varepsilon) \Big) \,. \end{aligned} \tag{2.14}$$

Next we argue as before and pass to the limit as $\varepsilon \to 0$, recalling that $F_\varepsilon(v_\varepsilon)$ is uniformly bounded. Thus we find $Q \in SO(d)$ and $\zeta \in \mathbb{R}^d$ such that

$$x = QU_i\,x + \zeta \quad \text{for } \mathcal{H}^{d-1}\text{-a.e. } x \in \Gamma\,. \tag{2.15}$$

Fix now $\bar{x} \in \Gamma$ such that (2.15) holds. For $\mathcal{H}^{d-1}$-a.e. $x \in \Gamma$, by (2.15) we have $x \in \bar{x} + \ker(QU_i - I)$. This implies

$$\mathcal{H}^{d-1}\Big(\Gamma \backslash \big(\bar{x} + \ker(QU_i - I)\big)\Big) = 0\,,$$

which contradicts the assumption (1.4). □

Remark 2.9. *If* (1.4) *is not satisfied, one can easily produce sequences of displacements such that* (1.18) *does not hold even if* $\mathcal{H}^{d-1}(\Gamma) > 0$. *More precisely, assume that* $\mathcal{H}^{d-1}(\Gamma) > 0$ *and that*

$$\exists\,\bar{\imath} \neq 1\,,\ \exists\,\bar{Q} \in SO(d)\,,\ \exists\,\bar{x} \in \mathbb{R}^d \colon \mathcal{H}^{d-1}\Big(\Gamma \backslash \big(\bar{x} + \ker(\bar{Q}U_{\bar{\imath}} - I)\big)\Big) = 0\,, \tag{2.16}$$

thus (1.4) *is violated. Define*

$$v_\varepsilon(x) := \bar{Q}U_{\bar{\imath}}(x - \bar{x}) + \bar{x} + \varepsilon\, g(x)\,.$$

Then v_ε *satisfies* (1.2) *by* (2.16)*, thus*

$$u_\varepsilon(x) = \frac{(\bar{Q}U_{\bar{\imath}} - I)(x - \bar{x})}{\varepsilon} + g(x)$$

satisfies (1.5). *However* $\nabla u_\varepsilon = \frac{(\bar{Q}U_{\bar{\imath}} - I)}{\varepsilon} + \nabla g$ *is not equibounded. This shows the necessity of assuming* (1.4) *in order to prove the compactness of the displacements.*

2.4. **Compactness with external forces.** In this section we consider the case when the functional $\mathcal{F}_\varepsilon$ defined in (1.10) is complemented by an external load and we prove the compactness results stated in Theorem 1.7 (ii), for $d > 1$, and in Theorem 1.6 (ii), for $d = 1$.

Before passing to the proof of these results, let us comment on the additional conditions we have imposed in this case. The assumption $q > 1$ is needed in order to define a duality between the loading term and the displacement. Moreover we have imposed a restriction on the scaling conditions (1.9c)–(1.9d), cf. (1.21). In the case of zero external load, (1.9c) reduces to $\eta(\varepsilon) \geq C\varepsilon$ for $r = 1^*$; for $r < 1^*$, even smaller values of $\eta(\varepsilon)$ are allowed by (1.9c). In contrast, in order to deal with external forces, for $r \leq 1^*$ we need the stronger condition $\eta(\varepsilon) \gg \varepsilon$. This is particularly relevant in dimension two, in which case $1^* = 2$. Examples 3.1 and 3.2 show that, in the case of external forces, for $\eta(\varepsilon) \leq \varepsilon$ it is not possible to establish a compactness result in $W^{1,r}(\Omega;\mathbb{R}^d)$ even for $r \leq 1^*$.

Proof of Theorem 1.7 (ii). We observe that by the Poincaré inequality

$$\mathcal{F}_\varepsilon(u_\varepsilon) \leq M + \mathcal{L}(u_\varepsilon) \leq M + C\,\|\nabla u_\varepsilon\|_{L^{r\wedge q}(\Omega;\mathbb{R}^{d\times d})} \leq M + C\,\|\nabla u_\varepsilon\|_{L^q(\Omega;\mathbb{R}^{d\times d})}\,. \tag{2.17}$$

Moreover, by assumption (iv) we get

$$\int_{\Omega\backslash\Omega_\varepsilon} |\nabla u_\varepsilon|^q \,\mathrm{d}x \leq \frac{C}{\varepsilon^q}\int_{\Omega\backslash\Omega_\varepsilon} \mathrm{dist}^q(\nabla v_\varepsilon, K)\,\mathrm{d}x \leq \frac{C}{\varepsilon^2}\int_{\Omega\backslash\Omega_\varepsilon} W(x, I + \nabla u_\varepsilon)\,\mathrm{d}x \leq C\mathcal{F}_\varepsilon(u_\varepsilon)\,,$$

where Ω_ε is as in (2.3). On the other hand, in Ω_ε we have $|\nabla v_\varepsilon| \leq C$, thus $|\nabla u_\varepsilon| \leq C/\varepsilon$, a.e. Therefore, for any $\kappa < 1$ there exists C_κ such that

$$\|\nabla u_\varepsilon\|^q_{L^q(\Omega;\mathbb{R}^{d\times d})} \leq \frac{C}{\varepsilon^q} + C\mathcal{F}_\varepsilon(u_\varepsilon) \leq \frac{C_\kappa}{\varepsilon^q} + \kappa\|\nabla u_\varepsilon\|^q_{L^q(\Omega;\mathbb{R}^{d\times d})}\,,$$

where in the last inequality we employed (2.17) in combination with Young's inequality. We obtain that $\|\nabla u_\varepsilon\|_{L^q(\Omega;\mathbb{R}^{d\times d})} \leq C/\varepsilon$ and, by using again (2.17), that

$$\mathcal{F}_\varepsilon(u_\varepsilon) \leq \frac{C}{\varepsilon}\,. \tag{2.18}$$

In particular, this allows us to apply Theorem 2.2.

Assume now $r > 1^*$. In order to prove (1.18), it is sufficient to follow verbatim the proof of Theorem 1.7 (i) and (1.18) follows from (2.13). The only difference is that here $F_\varepsilon(v_\varepsilon)$ may not be uniformly bounded; however, assumptions (1.8) guarantee that the right-hand side of (2.14) tends to zero. Indeed, in the case $r > 1^* \vee q$ we have $\theta = r/r_*$ and thus, by (2.18) and the fact that $r_* > 1$, we get

$$\varepsilon^r\Big((F_\varepsilon(v_\varepsilon))^\theta + (F_\varepsilon(v_\varepsilon))^{\frac{r}{2}} + \varepsilon^{2-r}F_\varepsilon(v_\varepsilon)\Big) \leq C\Big(\varepsilon^{r(1-\frac{1}{r_*})} + \varepsilon^{\frac{r}{2}} + \varepsilon\Big) = o(1)\,.$$

In the case $1^* < r \leq q$ we have $\theta = 1^*$ and thus by (2.18) we have

$$\varepsilon^r\Big((F_\varepsilon(v_\varepsilon))^\theta + (F_\varepsilon(v_\varepsilon))^{\frac{r}{2}} + \varepsilon^{2-r}F_\varepsilon(v_\varepsilon)\Big) \leq C\Big(\varepsilon^{r-1^*} + \varepsilon^{\frac{r}{2}} + \varepsilon\Big) = o(1)\,.$$

This proves (1.18), which in combination with (2.17) yields

$$\|\nabla u_\varepsilon\|^r_{L^r(\Omega;\mathbb{R}^{d\times d})} \leq C\left(1 + \|\nabla u_\varepsilon\|^\theta_{L^{r\wedge q}(\Omega;\mathbb{R}^{d\times d})}\right) \leq C\left(1 + \|\nabla u_\varepsilon\|^\theta_{L^r(\Omega;\mathbb{R}^{d\times d})}\right)\,. \tag{2.19}$$

Since $\theta < r$, (2.19) in combination with Young's inequality yields (1.20) for $r > 1^*$.

Finally, we prove (1.22) in the case $r = 1^*$. Recalling (2.8) and using (1.21), one can write

$$\int_\Omega \tilde{h}_\varepsilon^{1^*}\,\mathrm{d}x \leq C\frac{\varepsilon^{2\cdot 1^*}}{\eta^{1^*}}(F_\varepsilon(v_\varepsilon),\Omega\backslash\Omega_\varepsilon^{i_\varepsilon})^{1^*} \leq o(1)\,\varepsilon^{1^*}(F_\varepsilon(v_\varepsilon),\Omega\backslash\Omega_\varepsilon^{i_\varepsilon})^{1^*},$$

where $o(1) \to 0$ as $\varepsilon \to 0$. As in the previous case, we follow the proof of Theorem 1.7 (i). The above inequality ensures that the right-hand side of (2.14) tends again to zero, so that (1.22) follows from (2.13). Inequality (1.22) together with (2.17) gives

$$\|\nabla u_\varepsilon\|^{1^*}_{L^{1^*}(\Omega;\mathbb{R}^{d\times d})} \leq C\left(1 + o(1)\,\|\nabla u_\varepsilon\|^{1^*}_{L^{1^*\wedge q}(\Omega;\mathbb{R}^{d\times d})}\right) \leq C\left(1 + o(1)\,\|\nabla u_\varepsilon\|^{1^*}_{L^{1^*}(\Omega;\mathbb{R}^{d\times d})}\right)\,,$$

from which we readily deduce (1.20). All the results proven for $r = 1^*$ trivially extend to $r < 1^*$, since the scaling on η in (1.21) does not depend on r and since $(W^{1,r\wedge q}(\Omega;\mathbb{R}^d))^* \subset (W^{1,1^*}(\Omega;\mathbb{R}^d))^*$. □

Proof of Theorem 1.6 (ii). The arguments to prove (2.17) and (2.18) hold in every dimension. By (2.18) and (1.17), estimate (2.1) implies (1.12)–(1.14). Then (1.16) follows from (1.14) and (2.17) in combination with Young's inequality. □

2.5. **Γ-convergence.** In this section we prove Theorem 1.8. We first recall some basic facts about Γ-convergence. Fix a sequence $\varepsilon_j \to 0^+$. By Theorem 1.7, it is easy to see that, for every $M > 0$, the set $\bigcup_j\{u\colon \mathcal{F}_{\varepsilon_j}(u) \leq M\}$ is relatively compact in the weak topology of $W^{1,r}(\Omega;\mathbb{R}^d)$. Therefore, by applying [8, Propositions 7.7 and 8.10], we may characterise the Γ-limit of $\{\mathcal{F}_{\varepsilon_j}\}$ in terms of weakly converging sequences. We introduce the functionals

$$\mathcal{F}'(u) := \Gamma\text{-}\liminf \mathcal{F}_{\varepsilon_j} = \inf\{\liminf_{j\to+\infty}\mathcal{F}_{\varepsilon_j}(u_j)\colon u_j \rightharpoonup u \text{ in } W^{1,r}(\Omega;\mathbb{R}^d)\}\,,$$

$$\mathcal{F}''(u) := \Gamma\text{-}\limsup \mathcal{F}_{\varepsilon_j} = \inf\{\limsup_{j\to+\infty}\mathcal{F}_{\varepsilon_j}(u_j)\colon u_j \rightharpoonup u \text{ in } W^{1,r}(\Omega;\mathbb{R}^d)\}\,.$$

In order to prove Theorem 1.8, we will show that $\mathcal{F}''(u) \le \mathcal{F}(u) \le \mathcal{F}'(u)$ for every function $u \in H^1_{g,\Gamma}(\Omega;\mathbb{R}^d)$.

Proof of Theorem 1.8. Step 1: $\mathcal{F}(u) \le \mathcal{F}'(u)$. Let $u_j \rightharpoonup u$ in $W^{1,r}(\Omega;\mathbb{R}^d)$. Upon to passing to a subsequence, it is not restrictive to assume that $\mathcal{F}_{\varepsilon_j}(u_j)$ is uniformly bounded. We will indeed prove that

$$\liminf_{j\to+\infty} \frac{1}{\varepsilon_j^2}\int_\Omega W(x, I+\varepsilon_j\nabla u_j(x))\,\mathrm{d}x \ge \frac{1}{2}\int_\Omega D^2W(x,I)[e(u)]^2\,\mathrm{d}x\,. \tag{2.20}$$

We remark that, in order to prove (2.20), we have to follow a slightly different strategy than that used in the proof of the analogous result in [2, 9] for one-well potentials, since in our case we get a priori estimates that are in general weaker than those obtained in their analysis. Nevertheless our proof could be also adopted in those cases. Set

$$B_j := \left\{x\in\Omega\colon |\nabla u_j(x)| \le \varepsilon_j^{-\frac{1}{3}}\right\}. \tag{2.21}$$

By (1.18), we get

$$|\Omega\backslash B_j|\,\varepsilon_j^{-r/3} \le \int_\Omega |\nabla u_j|^r\,\mathrm{d}x \le C\,,$$

thus $|\Omega\backslash B_j| \to 0$ as $j\to+\infty$. Hence, setting

$$w_j := \chi_{B_j} e(u_j)\,,$$

we get that $w_j \rightharpoonup e(u)$ in $L^r(\Omega;\mathbb{R}^{d\times d})$. Now we use the fact that, by frame indifference, the energy density can be written as

$$W(x,F) = V(x, \tfrac{1}{2}(F^TF - I))\,,$$

where $V\colon \Omega\times\mathbb{R}^{d\times d}_{sym} \to [0,+\infty]$. Using the properties of the square root of positive definite matrices (see for example [14, Chapter 5, §3.11]), one can show that V inherits all the properties of W; precisely, $V(\cdot,\cdot)$ is $(\mathcal{L}\times\mathcal{B})$-measurable and, for a.e. $x\in\Omega$, $V(x,0)=0$, $DV(x,0)=0$, $V(x,\cdot)$ is of class C^2 in $I_\sigma := \{A\in\mathbb{R}^{d\times d}_{sym}\colon |A|<\sigma\}$, and the second derivatives are bounded by a constant independent of x. Moreover, from (1.1) it follows that

$$D^2W(x,I)[A]^2 = D^2V(x,0)[A]^2 \ge \lambda|A|^2 \quad \text{for every } A\in\mathbb{R}^{d\times d}_{sym}\,,\ \text{for a.e. } x\in\Omega\,. \tag{2.22}$$

Set, for $k\in\mathbb{N}$,

$$\omega_k(x) := \sup_{|A|\le\frac{1}{k}} |D^2V(x,A) - D^2V(x,0)|\,.$$

Note that (ω_k) is a nonincreasing sequence and converges to 0 a.e. as $k\to+\infty$. Hence, fixed $\delta>0$, the sequence of sets

$$C^k := \{x\in\Omega\colon \omega_k(x)\le\delta\} \tag{2.23}$$

is increasing with respect to inclusion and $|\Omega\backslash\cup_{k\in\mathbb{N}} C^k| = 0$. Now, given $x\in B_j^k := B_j\cap C^k$, by a Taylor expansion of $V(x,\cdot)$ about 0, we get

$$\begin{aligned} W(x, I+\varepsilon_j\nabla u_j(x)) &= V\Big(x, \varepsilon_j e(u_j)(x) + \varepsilon_j^2 C(u_j)(x)\Big)\\ &= \frac{1}{2}D^2V\Big(x, s_j\big(\varepsilon_j e(u_j)(x)+\varepsilon_j^2C(u_j)(x)\big)\Big)[\varepsilon_j e(u_j)(x)+\varepsilon_j^2 C(u_j)(x)]^2 \end{aligned} \tag{2.24}$$

for some $s_j \in (0,1)$, where $C(u_j)(x) := (\nabla u_j(x))^T \nabla u_j(x)$. Note that, fixed $k \in \mathbb{N}$, by the very definition of B_j, for j large enough

$$|\varepsilon_j e(u_j)(x) + \varepsilon_j^2 C(u_j)(x)| \leq \frac{1}{k} \quad \text{for every } x \in B_j\,.$$

Now, fix $t \in (0,1)$ and choose $\delta < (1-t)\lambda$ in (2.23). Then, by (2.22) and (2.24), for all $x \in B_j^k$ and for j large enough we get

$$\begin{aligned} W(x, I{+}\varepsilon_j \nabla u_j(x)) &\geq \tfrac{1}{2} D^2 V(x,0)[\varepsilon_j e(u_j)(x){+}\varepsilon_j^2 C(u_j)(x)]^2 - \tfrac{1}{2}\omega_k(x)|\varepsilon_j e(u_j)(x){+}\varepsilon_j^2 C(u_j)(x)|^2 \\ &\geq \tfrac{t}{2} D^2 V(x,0)[\varepsilon_j e(u_j)(x){+}\varepsilon_j^2 C(u_j)(x)]^2 \end{aligned}$$

and thus

$$\begin{aligned} \frac{1}{\varepsilon_j^2}\int_\Omega W(x, I{+}\varepsilon_j\nabla u_j(x))\,\mathrm{d}x &\geq \frac{1}{\varepsilon_j^2}\int_{B_j^k} W(x, I{+}\varepsilon_j\nabla u_j(x))\,\mathrm{d}x \\ &\geq \frac{t}{2}\int_{B_j^k} D^2V(x,0)[e(u_j)(x) + \varepsilon_j C(u_j)(x)]^2\,\mathrm{d}x\,. \end{aligned}$$

Since $|\varepsilon_j C(u_j)(x)| \leq \varepsilon_j^{\frac{1}{3}}$ on B_j, we get that the sequence of functions $\hat{w}_j := w_j + \varepsilon_j C(u_j)$ still converges to $e(u)$ weakly in $L^r(\Omega;\mathbb{R}^{d\times d})$. Hence, by the convexity of $A \mapsto D^2V(x,0)[A]^2$, we have

$$\liminf_{j\to+\infty} \frac{1}{\varepsilon_j^2}\int_\Omega W(x, I{+}\varepsilon_j\nabla u_j(x))\,\mathrm{d}x \geq \frac{t}{2}\int_{C^k} D^2V(x,0)[e(u)(x)]^2\,\mathrm{d}x\,.$$

Letting first $k \to +\infty$ and then $t \to 1$, we get (2.20).

Step 2: $\mathcal{F}(u) \geq \mathcal{F}''(u)$. By Proposition A.3 and the continuity of $\mathcal{F}$ in $H^1(\Omega;\mathbb{R}^d)$, it suffices to prove the inequality for $u \in W^{1,\infty}_{g,\Gamma}(\Omega;\mathbb{R}^d) \cap W^{2,p}(\Omega;\mathbb{R}^d)$. Fix then such a u and observe that, since $W(x,I) = 0$ and $DW(x,I) = 0$ for a.e. $x \in \Omega$, by assumption (ii) on W and the boundedness of ∇u we get

$$\lim_{j\to+\infty} \frac{1}{\varepsilon_j^2} W(x, I + \varepsilon_j \nabla u)) = \frac{1}{2} D^2 W(x,I)[e(u)]^2 \quad \text{for a.e. } x \in \Omega\,.$$

Assumptions (ii)-(iii) also imply that there exists $C > 0$ such that for j large enough

$$\frac{1}{\varepsilon_j^2} W(x, I + \varepsilon_j \nabla u) \leq C|\nabla u|^2 \quad \text{for a.e. } x \in \Omega\,.$$

Hence, by dominated convergence, we deduce

$$\mathcal{F}''(u) \leq \lim_{j\to+\infty} \mathcal{F}_{\varepsilon_j}(u) = \mathcal{F}(u)\,.$$

This concludes the proof. □

Remark 2.10. *Assumption* (1.9a) *has been used only in the proof of the* Γ-lim sup *inequality to guarantee that the second-order perturbation vanishes in the limit of $\varepsilon \to 0$. Here we briefly discuss what happens if* (1.9a) *does not hold.*

If $\eta(\varepsilon) \sim \varepsilon^{\frac{2}{p}-1}$, then the proof of the one-well lower bound (Theorem 2.2) can be carried out by setting in (2.6) $\widetilde{W} \equiv 1$ *and* $\beta = 1$ *(which corresponds to working with the Euclidean distance). In this case, compactness holds also for $r = p^*$, but the Γ-limit will contain the additional term* $\int |\nabla^2 u|^p\,\mathrm{d}x$.

In the case $\eta(\varepsilon) \gg \varepsilon^{\frac{2}{p}-1}$, the Γ-limit can be finite only on affine maps. The domain will be either the set $\{g\}$, if $g|_\Gamma$ is the restriction of an affine map, or the empty set otherwise.

We finally prove the strong convergence of recovery sequences.

Proof of Theorem 1.9. The proof closely follows the lines of the proof of [2, Theorem 2.5]. Here we only recall the main steps and highlight the points where some additional argument is needed.

Let $\{u_j\}$ be a recovery sequence for $u \in H^1_{g,\Gamma}(\Omega;\mathbb{R}^d)$. By the Urysohn property, it suffices to show that from any subsequence (not relabeled) we can extract a further subsequence $\{u_{j_n}\}$ such that $u_{j_n} \to u$ strongly in $W^{1,r}(\Omega;\mathbb{R}^d)$. To this end, let B_j and C^k be defined by (2.21) and (2.23), respectively. In the proof of Theorem 1.8 we have shown that for each $t \in (0,1)$ and $k \in \mathbb{N}$, choosing $\delta < (1-t)\lambda$ in the definition of C^k and setting $B^k_j = B_j \cap C^k$, we have

$$\begin{aligned}\liminf_{j\to+\infty} \mathcal{F}_{\varepsilon_j}(u_j) &\geq \liminf_{j\to+\infty} \frac{1}{\varepsilon_j^2}\int_{B^k_j} W(x, I+\varepsilon_j\nabla u_j)\,\mathrm{d}x \\ &\geq \frac{t}{2}\int_{B^k_j} D^2W(x,I)[e(u_j)(x)]^2\,\mathrm{d}x \geq \frac{t}{2}\int_{C^k} D^2W(x,I)[e(u)(x)]^2\,\mathrm{d}x\,.\end{aligned}$$

Since $\mathcal{F}_{\varepsilon_j}(u_j) \to \mathcal{F}(u)$, by a diagonal argument we can find sequences $j_n \to +\infty$, $k_n \to +\infty$, and $t_n \to 1$ such that, setting $\hat{B}_n := B^{k_n}_{j_n}$, we have

$$\begin{aligned}&\lim_{n\to+\infty} \frac{1}{\varepsilon_{j_n}}\int_{\hat{B}_n} W(x, I+\varepsilon_{j_n}u_{j_n}(x))\,\mathrm{d}x = \int_\Omega D^2W(x,I)[e(u)(x)]^2\,\mathrm{d}x\,, \\ &\lim_{n\to+\infty} \int_{\hat{B}_n} D^2W(x,I)[e(u_{j_n})(x)]^2\,\mathrm{d}x = \int_\Omega D^2W(x,I)[e(u)(x)]^2\,\mathrm{d}x\,, \\ &\lim_{n\to+\infty} |\Omega\backslash\hat{B}_n| = 0\,.\end{aligned} \tag{2.25}$$

The last two equalities above, together with the positive definiteness of $D^2W(x,I)$ on symmetric matrices and the weak convergence of $e(u_{j_n})$ to $e(u)$ in L^r, prove that $\chi_{\hat{B}_n}e(u_{j_n}) \to e(u)$ strongly in L^2. Hence, following the proof of [2, Theorem 2.5], the strong convergence of u_{j_n} to u in $W^{1,r}$ is a consequence of the following two properties:

(i) $\left\{\dfrac{1}{\varepsilon^r_{j_n}}\mathrm{dist}^r(I+\varepsilon_{j_n}\nabla u_{j_n}, SO(d))\right\}$ is equiintegrable,

(ii) $\{|\nabla u_{j_n}|^r\}$ is equiintegrable.

Once (i) is proven, the proof of (ii) follows verbatim that of [2, Theorem 2.5]. The proof of (i) can be performed in two steps. First, one proves that

$$\left\{\frac{1}{\varepsilon^r_{j_n}}\mathrm{dist}^r(I+\varepsilon_{j_n}\nabla u_{j_n}, SO(d))\chi_{\hat{B}_n}\right\} \text{ is equiintegrable,}$$

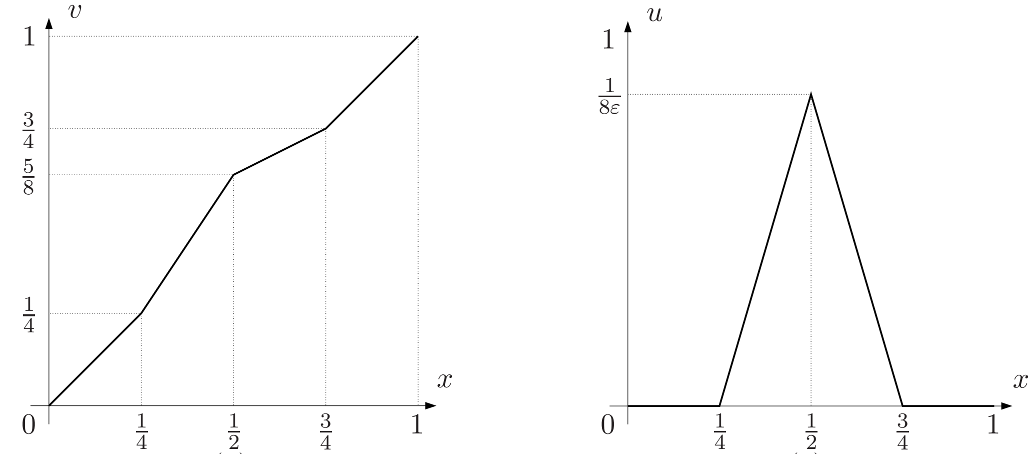
which can be done again as in [2, Theorem 2.5]. Second, one shows that

$$\frac{1}{\varepsilon^r_{j_n}}\int_{\Omega\backslash\hat{B}_n} \mathrm{dist}^r(I+\varepsilon_{j_n}u_{j_n}, SO(d))\,\mathrm{d}x \to 0\,. \tag{2.26}$$

To this end, note that, by (1.9a), (2.25), and the fact that $\mathcal{F}_{\varepsilon_j}(u_j) \to \mathcal{F}(u)$, we have

$$\lim_{n\to+\infty} \frac{1}{\varepsilon^2_{j_n}}\int_{\Omega\backslash\hat{B}_n} W(x, I+\varepsilon_{j_n}\nabla u_{j_n}(x))\,\mathrm{d}x + \eta^p(\varepsilon_{j_n})\varepsilon^{p-2}_{j_n}\int_\Omega |\nabla^2 u_{j_n}|^p\,\mathrm{d}x = 0\,.$$

Hence, using estimate (2.2a) with $A = \Omega\backslash\hat{B}_n$, we deduce (2.26). $\square$



(A) (B)

Figure 1. Example 3.1.

3. Optimality of the scaling

In the following examples we show that the compactness results proven above may not hold if the scaling assumptions (1.9b)–(1.9d) are not satisfied. We obtain counterexamples in the cases where the exponent r from (1.8) satisfies $r \leq 1^* \vee q$ or $r = 2$. In the case $1^* \vee q < r < 2$ (which is relevant for $d \geq 3$), the problem to find an optimal threshold for the compactness result is still open.

We assume that in the boundary conditions (1.2) and (1.5) the datum is $g = 0$ and that W does not depend on x. In all our examples, the energy wells are compatible, i.e., there are rank-one connections between them (see Remark 2.3).

Having in mind applications where the admissible deformations satisfy a non-interpenetration condition, we consider energy wells that are contained in a subset of matrices with positive determinant and we assume that W is bounded on such set. Nevertheless, our examples can be easily generalised.

We will employ a family of mollifiers $\rho_n \in C_c^\infty(\mathbb{R}^d; [0, +\infty))$ such that $\operatorname{supp} \rho_n \subset B_{\frac{1}{n}}(0)$, $\int_{\mathbb{R}^d} \rho_n \, \mathrm{d}x = 1$, and $|\nabla \rho_n| \leq C n^{d+1}$. Given $v \in W^{1,\infty}(\Omega; \mathbb{R}^d)$, we consider $v_n := \rho_n * v \in C^\infty(\Omega; \mathbb{R}^d)$, where $*$ denotes the convolution product. Observe that $\nabla v_n = \rho_n * \nabla v = \nabla \rho_n * v$ is uniformly bounded and that $\nabla^2 v_n = \nabla \rho_n * \nabla v$ satisfies $|\nabla^2 v_n| \leq Cn$.

Example 3.1. *The present one-dimensional example shows that, when $\eta(\varepsilon) \leq C\varepsilon^2$, sequences v_ε with equibounded energy $F_\varepsilon(v_\varepsilon)$ may display deformation gradients taking values in two different wells in sets with non vanishing measure. Moreover, a suitable load can be added so that compactness does not hold if $\eta(\varepsilon) \leq C\varepsilon$. This proves the optimality of the scaling* (1.9b) *for the problem without external forces (Theorem 1.6 (i)) and of the scaling* (1.17) *for the problem with external forces (Theorem 1.6 (ii)).*

Assume that $K = \{\frac{1}{2}, 1, \frac{3}{2}\}$ and let $\Omega = (0,1)$. Define $v \in W^{1,\infty}(\Omega)$ as follows:

$$v(x) := \begin{cases} x & \text{in } (0,1)\setminus(\frac{1}{4}, \frac{3}{4})\,, \\ \frac{3}{2}x - \frac{1}{8} & \text{in } (\frac{1}{4}, \frac{1}{2}]\,, \\ \frac{1}{2}x + \frac{3}{8} & \text{in } (\frac{1}{2}, \frac{3}{4})\,. \end{cases}$$

See Figure 1. The corresponding displacement is

$$u(x) := \begin{cases} 0 & \text{in } (0,1)\backslash(\frac{1}{4},\frac{3}{4})\,, \\ \frac{1}{\varepsilon}(\frac{1}{2}x - \frac{1}{8}) & \text{in } (\frac{1}{4},\frac{1}{2}]\,, \\ \frac{1}{\varepsilon}(-\frac{1}{2}x + \frac{3}{8}) & \text{in } (\frac{1}{2},\frac{3}{4})\,. \end{cases}$$

Consider approximating sequences $v_n := \rho_n * v$ *and* $u_n := \rho_n * u$ *as above. Then* u_n' *is not bounded in* $L^r(0,1)$ *for any* $r \geq 1$.

Note that $W(v_n')$ *and* v_n'' *are different from zero only in the intervals* $(\frac{1}{4}-\frac{1}{n},\frac{1}{4}+\frac{1}{n})$, $(\frac{1}{2}-\frac{1}{n},\frac{1}{2}+\frac{1}{n})$, *and* $(\frac{3}{4}-\frac{1}{n},\frac{3}{4}+\frac{1}{n})$, *where* $W(v_n')$ *is bounded. Therefore, taking into account the bound on* v_n'', *we obtain*

$$F_\varepsilon(v_n) \leq \frac{C}{\varepsilon^2 n}(1+\eta^p n^p)\,.$$

Choose now $n = n(\varepsilon) = [\eta^{-1}]$, *so*

$$F_\varepsilon(v_{n(\varepsilon)}) \leq C\frac{\eta}{\varepsilon^2}\,.$$

If $\eta(\varepsilon) \leq C\varepsilon^2$, *then* $F_\varepsilon(v_{n(\varepsilon)})$ *is equibounded. This shows that the assumption* (1.9b) *in Theorem 1.6 (i) is optimal.*

We now consider the case of non-zero external loads. Let $L > 0$ *and set*

$$\mathcal{L}(u) = L\int_0^1 u(x)\,\mathrm{d}x\,.$$

Then

$$\mathcal{L}(u_{n(\varepsilon)}) \sim \frac{L}{\varepsilon}\,.$$

If $\eta(\varepsilon) \leq C\varepsilon$, *then both the energetic and the loading term are unbounded. However,*

$$\mathcal{F}_\varepsilon(u_{n_\varepsilon}) - \mathcal{L}(u_{n_\varepsilon}) \leq \frac{C}{\varepsilon} - \frac{L}{\varepsilon}\,,$$

which is bounded from above whenever $L \geq C$. *This shows that the assumption* (1.17) *in Theorem 1.6 (ii) is optimal.*

Example 3.2. *In this second example, for* $d > 1$ *we show the optimality of the scaling assumption on* $\eta(\varepsilon)$ *in Theorem 1.7 (i) in the cases* $r \leq 1^* \vee q$ *and* $r = 2$. *This also implies the optimality of Theorem 1.7 (ii) in the cases* $1^* < r \leq q$ *and* $r = 2$, *since in the example we consider the special case of zero external load. Moreover, adding a suitable load, we show that compactness does not hold if* $\eta(\varepsilon) \leq C\varepsilon$, *proving the optimality of the scaling* (1.21) *in Theorem 1.7 (ii) in the case* $r = 1^*$.

Let $d > 1$ *and let* $U = I + e \otimes (1,,\dots 1)$, *where* $e \in \mathbb{R}^d$ *has norm small enough so that* $U \subset B_\rho(I) \subset \{A \in \mathbb{R}^{d\times d} :\ \det A > 0\}$, *for some* $\rho > 0$. *Assume that* $K = K_1 \cup K_2$, *with* $K_1 = SO(d)$ *and* $K_2 = SO(d)U$ *and that* $\eta(\varepsilon) \ll \varepsilon^{2-\frac{r}{1^*}}$. *Let* $\Omega = (0,1)^d$, *given* $\nu > 0$ *let* $\Omega_\nu = \Omega \cap \{x_1 + x_2 + \dots + x_d \leq \nu\}$ *and set* $\Gamma = \partial\Omega\backslash\overline{\Omega}_{\bar\nu}$ *for some* $\bar\nu \in (0,1)$, *see Figure 2. Define* $v^\nu \in W^{1,\infty}(\Omega;\mathbb{R}^d)$ *so that* $v^\nu(x) = x$ *in* $\Omega\backslash\Omega_\nu$ *and* $\nabla v^\nu = U$ *in* Ω_ν.

Define $v_n^\nu := \rho_n * v^\nu$ *as above. Note that* $W(\nabla v_n^\nu)$ *and* $\nabla^2 v_n^\nu$ *are supported in a* $\frac{1}{n}$*-neighbourhood of* $\Omega \cap \partial\Omega_\nu$ *whose volume is proportional to* $\frac{\nu^{d-1}}{n}$; *in such set* $W(\nabla v_n^\nu)$ *is bounded. Therefore,*

$$F_\varepsilon(v_n^\nu) \leq \frac{C}{\varepsilon^2}\frac{\nu^{d-1}}{n}(1+\eta^p n^p)\,.$$

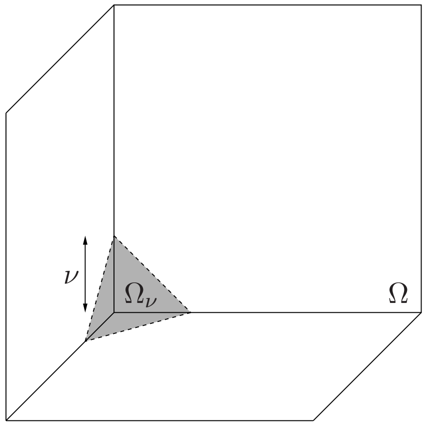


FIGURE 2. Example 3.2.

Now choose $n = n(\varepsilon) = [\eta^{-1}]$ *and* $\nu = (\varepsilon^2 n)^{\frac{1}{d-1}}$*, so that the energy is equibounded. Moreover we easily infer that for* ε *small enough* $u_{n(\varepsilon)} := \frac{1}{\varepsilon}(I - v^\nu_{n(\varepsilon)}) \in \mathcal{W}^{p,q}_0(\Omega;\mathbb{R}^d)$*. Since* $\eta(\varepsilon) \ll \varepsilon^{2-\frac{r}{1^*}}$*, we have*

$$\nu^d \geq C\Big(\frac{\varepsilon^2}{\eta}\Big)^{\frac{d}{d-1}} \gg \varepsilon^r\,.$$

This implies that $|\Omega_\nu| = C\nu^d \gg C\varepsilon^r$*. In particular, since* $|\nabla u_{n(\varepsilon)}| \sim 1/\varepsilon$ *on* Ω_ν*, the norm* $\|\nabla u_{n(\varepsilon)}\|_{L^r}$ *is unbounded in* ε*. This provides a counterexample to Theorem 1.7 (i) in the cases* $r \leq 1^* \vee q$ *and* $r = 2$ *(see Remark 1.3), as well as a counterexample to Theorem 1.7 (ii) in the cases* $1^* < r \leq q$ *in every dimension and* $r = 2$ *in dimension larger than* 2*. Therefore, in these cases the scaling* (1.9c) *is optimal.*

Let now $\eta \leq C\varepsilon$ *. Let* $L > 0$ *and set*

$$\mathcal{L}(u) = L\int_\Omega u(x)\,dx. \tag{3.1}$$

If in the definition of v^ν_n *we choose* $\nu = O(1)$*, we have that, as in Example 3.1, both the energetic and the loading term are unbounded and*

$$\mathcal{F}_\varepsilon(u_{n_\varepsilon}) - \mathcal{L}(u_{n_\varepsilon}) \leq \frac{C}{\varepsilon} - \frac{L}{\varepsilon}\,,$$

which is bounded from above whenever $L \geq C$*. In this case* $|\Omega_\nu| \geq C > 0$ *and, since* $|\nabla u_{n(\varepsilon)}| \sim 1/\varepsilon$ *on* Ω_ν*, the norm* $\|\nabla u_{n(\varepsilon)}\|_{L^r}$ *is unbounded in* ε *for every* $r \geq 1$*. This shows the optimality of the scaling* (1.21) *in Theorem 1.7 (ii) in the case* $r = 1^*$*. This also shows that if* $\eta \leq C\varepsilon$ *it is not possible to establish a compactness result in* $W^{1,r}(\Omega;\mathbb{R}^d)$ *even for* $r < 1^*$ *in the case of applied loads.*

Note that in the examples above we had to choose a Dirichlet boundary Γ strictly contained in $\partial\Omega$. However, if K consists of many compatible wells, it is possible to construct counterexamples with $\Gamma = \partial\Omega$. In the following two-dimensional example we consider the case of four compatible wells. The example can be easily generalised to dimension $d > 2$ with $d + 2$ compatible wells.

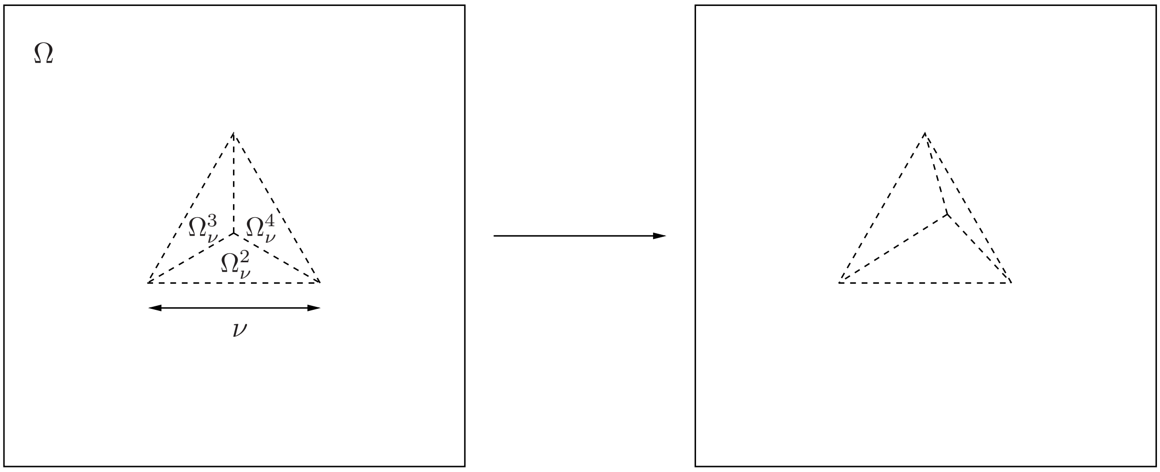


FIGURE 3. Example 3.3. The reference configuration Ω (left) and its image via v^ν (right).

Example 3.3. *Let $d = 2$. Assume that $K = \bigcup_{i=1}^4 K_i$ where $K_i = SO(d)U_i$, $U_1 = I$, and the matrices U_i for $i = 2, 3, 4$ are chosen in such a way that the following hold:*

$$U_2\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}1\\0\end{pmatrix}, \qquad U_2\begin{pmatrix}\frac12\\\frac{\sqrt3}{6}\end{pmatrix} = \begin{pmatrix}\frac12\\\frac{\sqrt3}{6}\end{pmatrix} + \begin{pmatrix}a\\b\end{pmatrix},$$
$$U_3\begin{pmatrix}\frac12\\\frac{\sqrt3}{2}\end{pmatrix} = \begin{pmatrix}\frac12\\\frac{\sqrt3}{2}\end{pmatrix}, \quad U_3\begin{pmatrix}\frac12\\\frac{\sqrt3}{6}\end{pmatrix} = \begin{pmatrix}\frac12\\\frac{\sqrt3}{6}\end{pmatrix} + \begin{pmatrix}a\\b\end{pmatrix},$$
$$U_4\begin{pmatrix}-\frac12\\\frac{\sqrt3}{2}\end{pmatrix} = \begin{pmatrix}-\frac12\\\frac{\sqrt3}{2}\end{pmatrix}, \quad U_3\begin{pmatrix}-\frac12\\\frac{\sqrt3}{6}\end{pmatrix} = \begin{pmatrix}-\frac12\\\frac{\sqrt3}{6}\end{pmatrix} + \begin{pmatrix}a\\b\end{pmatrix},$$

where $(a, b) \neq (0, 0)$ is a fixed vector. Note that the four wells are compatible.

Assume $\eta(\varepsilon) \ll \varepsilon^{2-\frac{r}{1^*}}$. Let $\Omega = (0, 1)^2$ and let $\Omega_\nu = \bigcup_{i=2}^4 \Omega_\nu^i$ be an equilateral triangle of side $\nu > 0$, centred at $(\frac12, \frac12)$, with one side parallel to $\binom{1}{0}$, where $(\Omega_\nu^i)_{i=2,3,4}$ is a partition of Ω_ν into three congruent triangles, numbered as in Figure 3. Let $\Gamma = \partial\Omega$. Define $v^\nu \in W^{1,\infty}(\Omega; \mathbb{R}^d)$ so that $v^\nu(x) = x$ in $\Omega\backslash\Omega_\nu$ and $\nabla v^\nu = U_i$ in Ω_ν^i for $i = 2, 3, 4$. Note that v^ν is a continuous piecewise affine map and $v^\nu(\frac12, \frac12) = (\frac12, \frac12) + (a, b)$.

Define $v_n^\nu := \rho_n * v^\nu$ as in Example 3.2 and choose $n = [\eta^{-1}]$ and $\nu = \varepsilon^2 n$. Then, arguing as in Example 3.2, it is possible to show that $F_\varepsilon(v_{n(\varepsilon)}^\nu)$ is uniformly bounded and $|\Omega_\nu| = C\nu^2 \gg C\varepsilon^r$. As above, it follows that the corresponding sequence of displacements is unbounded in $W^{1,r}_{\mathrm{loc}}(\Omega; \mathbb{R}^d)$. Analogously, if $\eta \leq C\varepsilon$ and $\mathcal{L}$ is defined as in (3.1), we can choose $L > 0$ so that $\mathcal{F}(u_{n(\varepsilon)}) - \mathcal{L}(u_{n(\varepsilon)})$ is bounded and the norm $\|\nabla u_n\|_{L^r}$ is unbounded in ε for every $r \geq 1$.

4. LINEARISATION IN A DISCRETE SETTING

In the present section we derive linear elasticity from a two-well discrete model. Our aim is to show that the role of the singular term in the continuum model studied before is played in this setting by interactions beyond nearest neighbours. We will indeed see that such interactions prevent too many jumps from one well to another. We focus on the simple but meaningful case of a two-dimensional discrete system governed by pairwise harmonic interactions between nearest and next-to-nearest neighbours and on a scaling regime that ensures compactness properties of the displacement fields in the weak topology of H^1. Under such assumptions, transitions between the wells may still take place, but they can only involve a finite number of atoms. This yields a great simplification, compared to the continuum setting, in the proof of compactness. The extension of this analysis to a broader class of interacting potentials and to more general scaling regimes will be provided in a subsequent paper.

Let $\mathrm{v}_1 = (1,0)$, $\mathrm{v}_2 = (\frac{1}{2}, \frac{\sqrt{3}}{2})$, $\mathrm{v}_3 = \mathrm{v}_2 - \mathrm{v}_1$ and $\mathcal{L}$ be the lattice on $\mathbb{Z}$ generated by v_1 and v_2, $\mathcal{L} := \{x_1\mathrm{v}_1 + x_2\mathrm{v}_2 \colon x_1, x_2 \in \mathbb{Z}\}$. For any $\eta > 0$, set $\mathcal{L}_\eta := \eta\mathcal{L}$. We denote by $\mathcal{T}_\eta$ the triangulation subordinated to $\mathcal{L}_\eta$, that is the collection of equilateral triangles with side η and vertices in $\mathcal{L}_\eta$.

We study the case where the particles have equal mass and are bonded by harmonic springs connecting nearest and next-to-nearest neighbours, that is, particles of the type x and $x + \eta\xi$ in $\mathcal{L}_\eta$, where ξ varies in the set $B = B_1 \cup B_2$, with

$$
\begin{aligned}
&B_1 = \{\pm\mathrm{v}_1, \pm\mathrm{v}_2, \pm\mathrm{v}_3\}\,,\\
&B_2 = \{\pm\mathrm{w}_1, \pm\mathrm{w}_2, \pm\mathrm{w}_3\}\,,\\
&\mathrm{w}_1 = \mathrm{v}_1 + \mathrm{v}_2\,,\ \ \mathrm{w}_2 = \mathrm{v}_2 + \mathrm{v}_3\,,\ \ \mathrm{w}_3 = \ \mathrm{v}_3 - \mathrm{v}_1\,.
\end{aligned}
$$

See Figure 4.

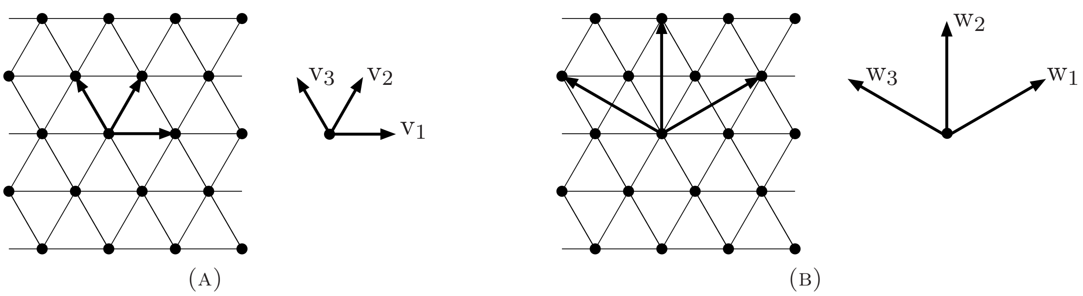


FIGURE 4. The lattice $\mathcal{L}$ and the vectors in B_1 (image A) and B_2 (image B).

We assume that the equilibrium length of the spring between x and $x + \eta\xi$ is $\eta|\xi|$ and that the elastic constants do not depend on x. Given a smooth bounded open subset Ω of $\mathbb{R}^2$, we consider the corresponding portion of lattice points of $\mathcal{L}_\eta$

$$\Omega_\eta := \{x \in \mathcal{L}_\eta \colon \exists\, y \in \Omega \cap \mathcal{L}_\eta \text{ such that } x \in y + \eta B\}\,.$$

Hence, the energy of a deformation $v \colon \Omega_\eta \to \mathbb{R}^2$ is given by

$$E^\eta(v) := \eta^2 \sum_{\substack{x \in \Omega \cap \mathcal{L}_n \\ \xi \in B}} \frac{1}{2} K_\xi \Big(\Big| \frac{v(x + \eta\xi) - v(x)}{\eta} \Big| - |\xi| \Big)^2, \tag{4.1}$$

where K_ξ are positive numbers. Note that

$$E^\eta(v) = E^\eta(Qv) \quad \text{for every } Q \in O(2)\,.$$

In particular E^η is frame-indifferent and admits as minimisers all isometries. Here we can regard $O(2)$ as the union of two rank-one connected wells,

$$O(2) = SO(2) \cup SO(2)J\,,$$

where $J = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. Moreover we underline that the scaling factor η^2 in (4.1) corresponds to a bulk scaling. In fact, using a general result in [3], the asymptotic behaviour as $\eta \to 0$ of E^η can be studied in terms of Γ-convergence and leads to a continuum limit described by an integral functional of the form $\int_\Omega f(\nabla u)\,dx$ on $H^1(\Omega;\mathbb{R}^2)$.

With in mind to perform an asymptotic analysis around an equilibrium position, we define the following rescaling of (4.1):

$$E^\eta_\varepsilon(v) := \frac{1}{\varepsilon^2} E^\eta(v),$$

and we express it in terms of the displacement $u(x) = \dfrac{v(x) - x}{\varepsilon}$

$$E^\eta_\varepsilon(x + \varepsilon u) = \frac{\eta^2}{\varepsilon^2} \sum_{\substack{x \in \Omega \cap \mathcal{L}_\eta \\ \xi \in B}} \frac{1}{2} K_\xi \Big(\Big| \xi + \varepsilon \frac{u(x + \eta\xi) - u(x)}{\eta} \Big| - |\xi| \Big)^2 .$$

In order to be consistent with the first part of the paper, we choose ε as the main parameter and $\eta(\varepsilon)$ as a function of ε.

In order to avoid technicalities, as a further simplification, we assume that the boundary datum g satisfies $g \in W^{1,\infty}(\mathbb{R}^2;\mathbb{R}^2) \cap C^2(\mathbb{R}^2;\mathbb{R}^2)$. Define $\mathcal{A}^g_\varepsilon(\Omega)$ as the class of all functions $u\colon \Omega_{\eta(\varepsilon)} \to \mathbb{R}^2$ that satisfy the condition $u(x) = g(x)$ at all boundary points $x \in \partial\,\Omega_{\eta(\varepsilon)}$, where

$$\partial\,\Omega_{\eta(\varepsilon)} := \{x \in \Omega_{\eta(\varepsilon)} \colon x + \eta(\varepsilon) B \not\subset \Omega\}.$$

For later convenience we identify each $u \in \mathcal{A}^g_\varepsilon(\Omega)$ with a function defined on the whole lattice $\mathcal{L}_{\eta(\varepsilon)}$ and coinciding with g at each point of $\mathcal{L}_{\eta(\varepsilon)} \backslash \Omega_{\eta(\varepsilon)}$. Abusing notation, we use the same letter to denote such extension and its piecewise affine interpolation with respect to the triangulation $\mathcal{T}_{\eta(\varepsilon)}$ subordinated to the lattice $\mathcal{L}_{\eta(\varepsilon)}$.

A second order Taylor expansion of $E^\eta_\varepsilon(x + \varepsilon Ax)$ with respect to ε about the point $\varepsilon = 0$ gives

$$E^\eta_\varepsilon(x + \varepsilon Ax) = \frac{2}{\sqrt{3}} |\Omega|\, \phi(A) + o_{\varepsilon,\eta}(1)\,,$$

where

$$\phi(A) := \sum_{\xi \in B} K_\xi \frac{(\xi^T A \xi)^2}{|\xi|^2} \quad \text{for every } A \in \mathbb{R}^{2\times 2}\,, \tag{4.2}$$

which turns out to be a quadratic form in the symmetric part $(A + A^T)/2$ of A (see, e.g., [5, Sec. 2] for details). The factor $\frac{2}{\sqrt{3}}$ comes from the fact that, since $\frac{\sqrt{3}}{2}$ is the area of the elementary cell of that lattice $\mathcal{L}$, it holds

$$\frac{2}{\sqrt{3}} = \lim_{\eta \to 0} \frac{\eta^2 \#(\Omega \cap \mathcal{L}_\eta)}{|\Omega|}\,,.$$

Set

$$\mathcal{E}_\varepsilon(u) := \begin{cases} E^{\eta(\varepsilon)}_\varepsilon(x + \varepsilon u) & \text{if } u \in \mathcal{A}^g_\varepsilon(\Omega)\,, \\ +\infty & \text{otherwise.} \end{cases} \tag{4.3}$$

Under the same assumptions (1.9a) and (1.9c) on the scaling of η for $p = q = r = d = 2$, which simply read

$$\lim_{\varepsilon \to 0} \eta(\varepsilon) = 0\,, \quad \eta(\varepsilon) \geq C\varepsilon\,, \tag{4.4}$$

we prove the following compactness and Γ-convergence results.

Theorem 4.1 (Compactness)**.** *Assume that $\eta(\varepsilon)$ satisfies* (4.4)*. If $u_\varepsilon \in \mathcal{A}^g_\varepsilon(\Omega)$ is a sequence such that $\mathcal{E}_\varepsilon(u_\varepsilon) \leq C$, then $\{u_\varepsilon\}$ is equibounded in $H^1(\Omega;\mathbb{R}^2)$. Furthermore,* $\det(I + \varepsilon\nabla u_\varepsilon) > 0$ *a.e. except in a number of triangles of $\mathcal{T}_{\eta(\varepsilon)}$ that is uniformly bounded in ε.*

Theorem 4.2 (Γ-convergence)**.** *Assume that $\eta(\varepsilon)$ satisfies* (4.4)*. As $\varepsilon \to 0$, the sequence* (4.3) *Γ-converges, with respect to the weak topology of $H^1(\Omega;\mathbb{R}^2)$, to the functional*

$$\mathcal{E}(u) := \begin{cases} \dfrac{2}{\sqrt{3}} \displaystyle\int_\Omega \phi(e(u))\,\mathrm{d}x & \text{if } u \in H^1_g(\Omega;\mathbb{R}^2)\,, \\ +\infty & \text{otherwise.} \end{cases}$$

Example 4.3 (Optimality of the scaling)**.** *If $\eta \ll \varepsilon$, then the conclusion of Theorem 4.1 does not hold. Indeed, let $\Omega = (0,1)^2$, Ω^i_ν, $i = 2,3,4$, $v^\nu \in W^{1,\infty}(\Omega;\mathbb{R}^2)$ be as in Example 3.3 and let $\nu = \nu(\varepsilon) \sim \frac{\varepsilon^2}{\eta}$. Define $v_\varepsilon : \Omega_\eta \to \mathbb{R}^2$ as $v_\varepsilon(x) := v^\nu(x)$ for $x \in \Omega_\eta$ and set $u_\varepsilon := \frac{1}{\varepsilon}(I - v)$. One can easily see that $u_\varepsilon \in \mathcal{A}^0_\varepsilon(\Omega)$ and that the only interactions giving a positive contribution to the energy are those crossing the boundaries of Ω^i_ν for $i = 2,3,4$, such contribution being uniformly bounded in ε. Hence $\mathcal{E}_\varepsilon(u_\varepsilon)$ is uniformly bounded. On the other hand, since $\nabla u_\varepsilon \sim \frac{1}{\varepsilon}$ on $\Omega_\nu = \cup_{i=2}^3 \Omega^i_\nu$ and $|\Omega_\nu| \gg \varepsilon^2$, we infer that $\|\nabla u_\varepsilon\|_{H^1(\Omega;\mathbb{R}^2)}$ is not uniformly bounded.*

The proof of Theorems 4.1 and 4.2 will follow after two technical lemmas. We introduce the energy of a single triangle $T \in \mathcal{T}_1$ with vertices $x_0, x_1, x_2 \in \mathcal{L}_1 = \mathcal{L}$,

$$E_{\mathrm{cell}}(v_F;T) := \sum_{i\leq j=0}^{2} \Big||F(x_i - x_j)| - 1\Big|^2 \quad \text{for every } F \in \mathbb{R}^{2\times 2}\,,$$

where v_F is the affine transformation $v_F(x) := Fx$. The following lemma gives a lower bound on $E_{\mathrm{cell}}(v_F;T)$ in terms of $\mathrm{dist}(F, O(2))$. It will be used in combination with Theorem 2.8.

Lemma 4.4. [4, Lemma 2.2] *There exists a constant $C > 0$ such that*

$$\mathrm{dist}^2(F, SO(2)) \leq C\, E_{\mathrm{cell}}(v_F;T) \quad \textit{for every } F \in \mathbb{R}^{2\times 2} \textit{ with } \det F \geq 0\,,$$
$$\mathrm{dist}^2(F, (O(2)\backslash SO(2))) \leq C\, E_{\mathrm{cell}}(v_F;T) \quad \textit{for every } F \in \mathbb{R}^{2\times 2} \textit{ with } \det F \leq 0\,.$$

In the next lemma one considers two neighbouring triangles (i.e., with a common side) where $\det\nabla v$ changes sign; then one sees that the energy localised in those two triangles is at least a positive constant. This highlights the role of next-to-nearest neighbour interactions in penalising transitions between the two wells $SO(2)$ and $SO(2)J$. We introduce the energy localised in two neighbouring triangles $T = [x_0, x_1, x_2]$, $S = [y_0, x_1, x_2]$ (see Figure 5) under the deformation $v : \mathcal{L}_1 \to \mathbb{R}^2$,

$$\begin{aligned} E_{\mathrm{cell}}(v; S\cup T) := \sum_{i\leq j=0}^{2} \Big||v(x_i) - v(x_j)| - |(x_i - x_j)|\Big|^2 + \sum_{j=1}^{2} \Big||v(y_0) - v(x_j)| - |(y_0 - x_j)|\Big|^2 \\ + \Big||v(y_0) - v(x_0)| - |(y_0 - x_0)|\Big|^2. \end{aligned}$$

Lemma 4.5. [4, Lemma 2.3] *There exists a positive constant C_0 with the following property: if two neighbouring triangles $S, T \in \mathcal{T}$ have different orientations in the deformed configuration, i.e.,*

$$\det(\nabla v|_S)\det(\nabla v|_T) \leq 0\,,$$

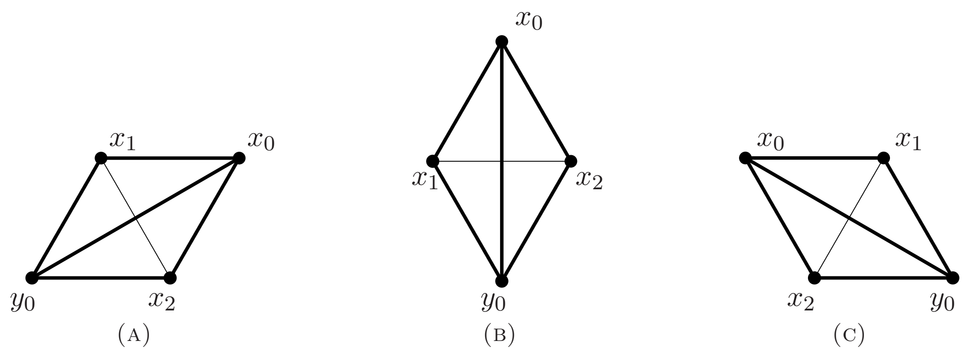


FIGURE 5. Neighbouring triangles in the lattice $\mathcal{L}$.

then $E_{\mathrm{cell}}(v; S \cup T) \geq C_0$.

Proof of Theorem 4.1. Let $v_\varepsilon(x) := x + \varepsilon u_\varepsilon(x)$ and recall that u_ε is identified with its piecewise affine interpolation. By assumption one finds that

$$\sum_{\substack{x\in\Omega_\eta\\ \xi\in B}} \frac{1}{2} K_\xi \Big(\Big| \frac{v_\varepsilon(x+\eta\xi) - v_\varepsilon(x)}{\eta} \Big| - |\xi| \Big)^2 \leq C \frac{\varepsilon^2}{\eta^2} \leq C' . \tag{4.5}$$

The above inequality implies that ∇v_ε is uniformly bounded in $L^\infty(\Omega)$. Furthermore, from (4.5) and Lemma 4.5 it follows that the number of neighbouring triangles of $\mathcal{T}_\eta$ intersecting Ω where $\det \nabla v_\varepsilon$ changes its sign is bounded by $\frac{C}{C_0}\frac{\varepsilon^2}{\eta^2} \leq C'$. On the other hand, the boundary condition on u_ε ensures that $\{x \in \Omega \colon \det \nabla v_\varepsilon > 0\}$ contains a number of triangles of order at least $1/\eta$. Therefore, the set $\{x \in \Omega \colon \det \nabla v_\varepsilon \leq 0\}$ consists of a uniformly bounded number of triangles and

$$|\{x \in \Omega \colon \det \nabla v_\varepsilon \leq 0\}| \leq C\varepsilon^2 . \tag{4.6}$$

Using Lemma 4.4, the uniform bound $\|\nabla v_\varepsilon\|_{L^\infty(\Omega;\mathbb{R}^{2\times 2})} \leq C$, and (4.6), we infer that

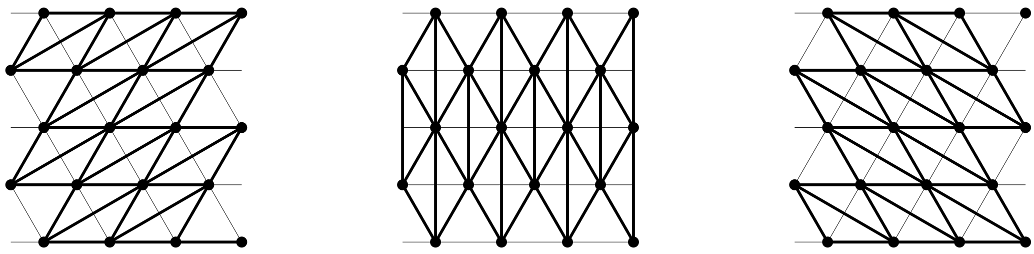
$$\int_\Omega \mathrm{dist}^2(\nabla v_\varepsilon; SO(2))\,\mathrm{d}x \leq C\varepsilon^2 .$$

We can now proceed as in the proof of Theorem 1.7 and deduce that $\|\nabla v_\varepsilon - I\|_{L^2(\Omega)} \leq C\varepsilon^2$, which, together with the boundary conditions satisfied by u_ε, yields the desired compactness. $\square$

Proof of Theorem 4.2. As usual, we prove a lower and an upper bound.

Step 1: Lower bound. By Proposition 4.1, we easily reduce our analysis to the case of deformations v_ε satisfying the constraint $\det \nabla v_\varepsilon > 0$, as outlined below. Under this restriction, a derivation of linear elasticity by Γ-convergence has been provided in [5] (see also [17]). Nevertheless, in order to exploit those results, an additional step is needed to bring back our analysis to the case of functionals that satisfy exactly the assumptions of the main theorem in [5]. To this end, we first write the discrete energy (4.1) as the sum of three contributions

$$E^\eta = \mathcal{I}^{\eta,1} + \mathcal{I}^{\eta,2} + \mathcal{I}^{\eta,3} ,$$



(A) (B) (C)

FIGURE 6. The energies $\mathcal{I}^{\eta,i}$, $i = 1, 2, 3$, in the proof of Theorem 4.2 account only for interactions between pairs of vertices of triangles in the triangulations marked with bold lines in (A), (B) and (C), respectively.

where each $\mathcal{I}^{\eta,i}$ only accounts for two interactions in B_2. Specifically, we set

$$\mathcal{I}^{\eta,i}(v):=\eta^2\Bigg[\frac{1}{2}\sum_{\substack{x\in\mathcal{L}_\eta\cap\Omega\\ \xi\in B_1^i}}\frac{1}{2}K_\xi\Big(\Big|\frac{v(x+\eta\xi)-v(x)}{\eta}\Big|-|\xi|\Big)^2+\sum_{\substack{x\in\mathcal{L}_\eta\cap\Omega\\ \xi=\pm\mathrm{w}_i}}\frac{1}{2}K_\xi\Big(\Big|\frac{v(x+\eta\xi)-v(x)}{\eta}\Big|-|\xi|\Big)^2\Bigg],$$

where

$$B_1^1 := \{\pm\mathrm{v}_1, \pm\mathrm{v}_2\}\,, \quad B_1^2 := \{\pm\mathrm{v}_2, \pm\mathrm{v}_3\}\,, \quad B_1^3 := \{\pm\mathrm{v}_3, \pm\mathrm{v}_1\}\,.$$

Note that each $\mathcal{I}^{\eta,i}$ only accounts for interactions between points of the lattice that are vertices of a suitable triangulation of $\mathbb{R}^2$ and hence it belongs to the class of discrete energies considered in [5] (see Figures 5 and 6).

Accordingly we can write

$$\mathcal{E}_\varepsilon(u) = \sum_{i=1}^{3}\frac{1}{\varepsilon^2}\mathcal{I}^{\eta(\varepsilon),i}(x+\varepsilon u)\,.$$

Now let $u_\varepsilon \in \mathcal{A}^g_\varepsilon(\Omega)$ converge to $u \in H^1_g(\Omega;\mathbb{R}^2)$ weakly in $H^1(\Omega;\mathbb{R}^2)$. We may suppose that $\mathcal{E}_\varepsilon(u_\varepsilon) \le C$. Proposition 4.1 ensures that the set $\{\det\nabla v_\varepsilon < 0\}$ consists of a uniformly bounded number of triangles. Therefore, there exist $x_1,\dots,x_m \in \Omega$ such that, up to subsequences, $\det\nabla v_\varepsilon > 0$ a.e. in $\Omega\setminus\cup_{i=1}^m B_\rho(x_i)$, for each $\rho > 0$ and ε sufficiently small (with $B_\rho(x_i)$ denoting the ball of radius ρ and center x_i). This implies that the sequence of functionals $\frac{1}{\varepsilon^2}\mathcal{I}^{\eta(\varepsilon),i}$ localised on $\Omega\setminus\cup_{i=1}^m B_\rho(x_i)$ satisfy along the sequence $\Big\{u_\varepsilon\Big|_{\Omega\setminus\cup_{i=1}^m B_\rho(x_i)}\Big\}$ all the assumptions of [5, Theorem 3.2], which in turn yields

$$\liminf_{\varepsilon\to0}\mathcal{E}_\varepsilon(u_\varepsilon) \ge \sum_{i=1}^{3}\liminf_{\varepsilon\to0}\frac{1}{\varepsilon^2}\mathcal{I}^{\eta(\varepsilon),i}(x+\varepsilon u_\varepsilon) \ge \frac{1}{\sqrt{3}}\sum_{i=1}^{3}\int_{\Omega\setminus\cup_{i=1}^m B_\rho(x_i)}\phi^i(e(u))\,\mathrm{d}x\,. \tag{4.7}$$

In the above formula the functions ϕ^i are given by

$$\phi^i(A) := \frac{1}{2}\sum_{\xi\in B_1^i}K_\xi\frac{(\xi^TA\xi)^2}{|\xi|^2}+\sum_{\xi=\pm\mathrm{w}_i}K_\xi\frac{(\xi^TA\xi)^2}{|\xi|^2} \qquad \text{for every } A\in\mathbb{R}^{2\times2}\,.$$

A straightforward computation shows that $\displaystyle\sum_{i=1}^{3} \phi^i(A) = \phi(A)$, where ϕ is defined in (4.2). Finally, taking in (4.7) the supremum with respect to ρ gives

$$\liminf_{\varepsilon \to 0} \mathcal{E}_\varepsilon(u_\varepsilon) \geq \mathcal{E}(u)\,.$$

Step 2: Upper bound. In order to complete the proof of Theorem 4.2 we need to show that the lower bound is actually attained, namely, for each $u \in H^1_g(\Omega;\mathbb{R}^2)$ there exists a sequence $u_\varepsilon \in \mathcal{A}^g_\varepsilon(\Omega)$ such that $u_\varepsilon \rightharpoonup u$ weakly in $H^1(\Omega;\mathbb{R}^2)$ and $\mathcal{E}_\varepsilon(u_\varepsilon) \to \mathcal{E}(u)$. Assume first that u also satisfies $u \in W^{1,\infty}(\mathbb{R}^2;\mathbb{R}^2) \cap C^2(\mathbb{R}^2;\mathbb{R}^2)$. Define u_ε as the piecewise affine interpolation of the values of u on $\Omega_{\eta(\varepsilon)} \backslash \partial\Omega_{\eta(\varepsilon)}$ and of g at all the other points of $\mathcal{L}_{\eta(\varepsilon)}$.

Let $x \in \Omega_\eta$ and $\xi \in B$ and set

$$D^\xi_\eta u_\varepsilon(x) := \frac{u(x+\eta\xi) - u(\xi)}{\eta}\,.$$

By a second order Taylor expansion with respect to ε of the energy corresponding to the interaction between x and $x + \eta\xi$, taking into account the equi-boundedness of ∇u_ε, we obtain

$$\frac{1}{2\varepsilon^2} K_\xi \Big(\Big| \xi + \varepsilon D^\xi_\eta u_\varepsilon(x) \Big| - |\xi| \Big)^2 = K_\xi \frac{\Big(D^\xi_\eta u_\varepsilon(x) \cdot \xi \Big)^2}{|\xi|^2} + o(\varepsilon)\,.$$

By the C^2 assumption on u, we get

$$\frac{1}{2\varepsilon^2} K_\xi \Big(\Big| \xi + \varepsilon \frac{u(x+\eta\xi) - u(x)}{\eta} \Big| - |\xi| \Big)^2 = K_\xi \frac{\Big(\xi^T \nabla u(x) \xi \Big)^2}{|\xi|^2} + o(1) + o(\varepsilon)\,.$$

Hence, summing up in ξ and x and letting $\varepsilon \to 0$, we get

$$\lim_{\varepsilon \to 0} \mathcal{E}_\varepsilon(u_\varepsilon) = \frac{2}{\sqrt{3}} \int_\Omega \phi(e(u))\mathrm{d}x\,. \tag{4.8}$$

In the general case $u \in H^1_g(\Omega;\mathbb{R}^2)$, we can find a sequence (u_k) in $g + C^\infty_c(\Omega;\mathbb{R}^2)$ such that $u_k \to u$ strongly in $H^1(\Omega;\mathbb{R}^2)$, so $\lim_{k\to+\infty} \int_\Omega \phi(e(u_k))\mathrm{d}x = \int_\Omega \phi(e(u))\mathrm{d}x$. Then, we conclude by (4.8) applied to u_k and by a standard diagonal argument. □

Appendix A

We recall the definition and the basic properties of capacity and we prove some related lemmas. Moreover we show that assumption (1.3) can be avoided, provided the exponents p and q defined in Section 1 are sufficiently large.

A.1. **Capacity and q.e.-equality.** We fix once and for all a bounded set $U \subset \mathbb{R}^d$ such that $\overline{\Omega} \subset U$. Given an arbitrary set $E \subset U$, the *capacity* of E is defined as

$$\mathrm{cap}(E) := \inf\left\{ \int_U |\nabla u|^2\,\mathrm{d}x \colon u \in H^1_0(U)\,,\ u \geq 1 \text{ a.e. in a neighbourhood of } E \right\}.$$

A property is said to hold cap-*quasi-everywhere* (q.e.) on E if it is satisfied on E except on a set of capacity zero. A function $u\colon E \to \overline{\mathbb{R}}$ is said to be cap-*quasicontinuous* on E if for every $\tau > 0$ there is an open set A_τ such that $u|_{E\backslash A_\tau}$ is continuous on $E \backslash A_\tau$ and $\mathrm{cap}(A_\tau) < \tau$. The notions of quasi-everywhere and quasicontinuity are independent of the choice of U.

Given $u \in H^1(\Omega)$, there exists a cap-quasicontinuous function on $\overline{\Omega}$ that coincides $\mathcal{L}^d$-a.e. with u. This function is uniquely defined up to sets of capacity zero. It is called the cap-*quasicontinuous representative* of u and is denoted by $\tilde{u}$. Moreover it satisfies

$$\lim_{\rho\to 0}\frac{1}{B_\rho(x)\cap\Omega}\int_{B_\rho(x)\cap\Omega}|u(y)-\tilde{u}(x)|\,\mathrm{d}y = 0 \quad \text{for cap-q.e. } x \in \overline{\Omega}\,. \tag{A.1}$$

If $u_n \to u$ strongly in $H^1(\Omega)$, then $\tilde{u}_n$ converges to $\tilde{u}$ cap-q.e. in $\overline{\Omega}$, up to subsequences. These properties can be found in [11, Section 4.8] and [13, Chapter 4] for the interior part. For the points on $\partial\Omega$ they can be easily obtained using an extension operator for the Lipschitz domain Ω.

By using assumption (1.3), we show that the boundary condition $u \in H^1_{g,\Gamma}(\Omega;\mathbb{R}^d)$ in (1.23) can be expressed by means of an equality q.e. of the quasicontinuous representatives. Without loss of generality we consider scalar functions with a homogeneous boundary condition.

Lemma A.1. *Let $u \in H^1_{0,\Gamma}(\Omega)$. Then $\tilde{u} = 0$ cap-q.e. on $\overline{\Gamma}$, where $\tilde{u}$ is the cap-quasicontinuous representative of u.*

Proof. Since Γ is open in the relative topology of $\partial\Omega$, there is an open set $\Omega_D \subset \mathbb{R}^d$ such that $\Gamma = \partial\Omega \cap \Omega_D$. We set

$$v = \begin{cases} u & \text{in } \Omega_D \cap \Omega\,, \\ 0 & \text{in } \Omega_D \backslash \Omega\,. \end{cases}$$

Then $v \in H^1(\Omega_D)$ because of the assumption $u \in H^1_{0,\Gamma}(\Omega)$. Let $\tilde{v}$ be the cap-quasicontinuous representative of v. By (A.1), for cap-q.e. $x \in \Gamma$ we have

$$\lim_{\rho\to 0}\int_{B_\rho(x)\backslash\Omega}|v(y)-\tilde{v}(x)|\,\mathrm{d}y \le \lim_{\rho\to 0}\int_{B_\rho(x)}|v(y)-\tilde{v}(x)|\,\mathrm{d}y = 0\,,$$

(with $B_\rho(x)$ denoting the ball of radius ρ and center x). Since $v = 0$ in $\Omega_D\backslash\Omega$, it turns out that $\tilde{v} = 0$ cap-q.e. on $\Gamma \cap \Omega_D = \Gamma$. By (1.3) it follows that $\tilde{v} = 0$ cap-q.e. on $\overline{\Gamma}$. The conclusion follows by the observation that $\tilde{u} = \tilde{v}$ on $\overline{\Gamma}$. □

A.2. **A density result.** We prove an approximation lemma.

Lemma A.2. *Let U be an open bounded subset of $\mathbb{R}^d$ and K be a compact subset of U. Let $u \in H^1(U)$ be such that $\tilde{u} = 0$ cap-q.e. on K, where $\tilde{u}$ is the quasicontinuous representative of u. Then there is a sequence $u_n \in C^\infty(U)$ such that $u_n = 0$ in a neighbourhood of K and u_n converges to u strongly in $H^1(U)$.*

Proof. Using a sequence of truncations, up to a diagonal argument we may assume that u is bounded. Moreover, without loss of generality we may assume that u is positive; the general case is solved by approximating the positive and the negative part of u. We thus restrict to the case where $0 \le u \le 1$.

By quasicontinuity of $\tilde{u}$, for every $n \in \mathbb{N}$ there is an open set $A_n \subset U$ such that $\tilde{u}|_{U\backslash A_n}$ is continuous in $U\backslash A_n$ and $\mathrm{cap}(A_n) < 1/n$. Let us denote by w_{A_n} the solution of the problem

$$w_{A_n} = \operatorname{argmin}\left\{\int_U |\nabla w|^2\,\mathrm{d}x \colon w \in H^1_0(U)\,,\ w = 1 \text{ a.e. in } A_n\right\}.$$

Since $\mathrm{cap}(A_n) < 1/n$, by the Poincaré inequality we have $w_{A_n} \to 0$ in $H^1(U)$ as $n \to +\infty$.

We define

$$v_n := \big(\tilde{u} - \tfrac{1}{n}\big)^+ \wedge (1 - w_{A_n})\,.$$

Since $0 \leq u \leq 1$, we have $v_n \to \tilde{u}$ strongly in $H^1(U)$. Moreover, $v_n = 0$ a.e. in $\{\tilde{u} < \frac{1}{n}\} \cup A_n$, which is an open set containing K. Finally we define u_n by regularising v_n e.g. by convolution, using the fact that $K \subset U$. □

We next prove the density property employed in the proof of Theorem 1.8 (Step 2).

Proposition A.3. *Let $p \in [1, +\infty)$ and let $g \in W^{1,\infty}(\Omega; \mathbb{R}^d) \cap W^{2,p}(\Omega; \mathbb{R}^d)$. Then $H^1_{g,\Gamma}(\Omega; \mathbb{R}^d)$ is the closure of $W^{1,\infty}_{g,\Gamma}(\Omega; \mathbb{R}^d) \cap W^{2,p}(\Omega; \mathbb{R}^d)$ in $H^1(\Omega; \mathbb{R}^d)$.*

Proof. Without loss of generality, we may assume that $g = 0$. Let $u \in H^1_{0,\Gamma}(\Omega; \mathbb{R}^d)$ and let $\tilde{u}$ be the quasicontinuous representative of u. By Lemma A.1 below, we have $\tilde{u} = 0$ cap-q.e. on $\overline{\Gamma}$. We can then apply Lemma A.2 to each component of $u = (u_1, \dots, u_d)$ choosing $K = \overline{\Gamma}$ and U an open bounded subset of $\mathbb{R}^d$ with $\overline{\Omega} \subset U$. The conclusion readily follows. □

Remark A.4. *Proposition A.3 can be proven by adapting the arguments of* [2, Proposition A.2] *if it is assumed in addition that Γ has Lipschitz boundary in $\partial\Omega$, cf.* [2, Definition 2.1] *for the notion of subset with Lipschitz boundary in $\partial\Omega$.*

We chose to give a different proof of the approximation property that requires weaker assumptions on the boundary of Γ in $\partial\Omega$, cf. (1.3).

Remark A.5. *If* (1.3) *does not hold, we prescribe the boundary condition in the following form, which is stronger than* (1.5)*:*

$$\tilde{u} = \tilde{g} \quad \text{cap-}q.e. \text{ on } \overline{\Gamma}\,,$$

where $\tilde{v}$ and $\tilde{g}$ are the cap-*quasicontinuous representatives of v and g, respectively. Then Theorem 1.8 is still true, provided one assumes $p > 2_*$ and $r = 2$, cf.* (1.8d). *This follows by adapting the proof outlined above and by taking into account the following property, which is a consequence of* (A.1)*: if $u_\varepsilon \in H^1(\Omega; \mathbb{R}^d)$ is such that $\tilde{u}_\varepsilon = \tilde{g}$* cap-*q.e. on an arbitrary set $E \subset \overline{\Omega}$ and u_ε converges to u weakly in $H^1(\Omega; \mathbb{R}^d)$, then $\tilde{u} = \tilde{g}$* cap-*q.e. on E.*

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