Simple proofs of some results of Reshetnyak

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Abstract

In this note, we give simpler proofs of the classical continuity and lower semicontinuity theorems of Reshetnyak.

1 Main Result

In 1968, Reshetnyak [20] proved two important results concerning the continuity and lower semicontinuity of functionals with respect to weak-star convergence of measures. These theorems are used in a variety of areas in the calculus of variations, ranging from problems in relaxation ([1],[3],[4],[6]) and estimates in Gamma convergence ([18]),[17],[19]), to anisotropic surface energies studied in continuum mechanics ([14],[9],[10],[11]) and beyond ([2],[7],[12]).

For X a locally compact, separable metric space, let $[M_b(X)]^m$ denote the space of $\mathbb{R}^m$-valued measures on X with finite total mass. Given $\mu \in [M_b(X)]^m$, we write $|\mu|$ for the total variation of μ and $\frac{d\mu}{d|\mu|}$ for the Radon-Nikodym derivative of μ with respect to $|\mu|$. Under these assumptions (see Proposition 1.43 and Remark 1.57 of [5]), we have that $[M_b(X)]^m$ is the dual of $[C_0(X)]^m$. Thus, for $\mu_n, \mu \in [M_b(X)]^m$, we have that $\mu_n \overset{*}{\rightharpoonup} \mu$ if

$$\lim_{n\to\infty} \int_X \phi \cdot d\mu_n \to \int_X \phi \cdot d\mu$$

for every $\phi \in [C_0(X)]^m$.

In [20], the following theorems are given.

Theorem 1.1 *Let X be a locally compact, separable metric space and μ_n, $\mu \in [M_b(X)]^m$. Assume that $\mu_n \overset{*}{\rightharpoonup} \mu$ and that*

$$\lim_{n\to\infty} \int_X g\left(x, \frac{d\mu_n}{d|\mu_n|}(x)\right) \, d|\mu_n| = \int_X g\left(x, \frac{d\mu}{d|\mu|}(x)\right) \, d|\mu| \tag{1.1}$$

for every continuous function $g : X \times \mathbb{R}^m \to \mathbb{R}$, positively 1-homogeneous and strictly convex in the second variable, satisfying the growth condition $|g(x,z)| \leq C|z|$ for each $(x,z) \in X \times \mathbb{R}^m$ and for some $C > 0$. Then

$$\lim_{n\to\infty} \int_X f\left(x, \frac{d\mu_n}{d|\mu_n|}(x)\right) \, d|\mu_n| = \int_X f\left(x, \frac{d\mu}{d|\mu|}(x)\right) \, d|\mu| \tag{1.2}$$

for every continuous function $f : X \times \mathbb{R}^m \to \mathbb{R}$ satisfying the growth condition $|f(x,z)| \leq C_1|z|$ for each $(x,z) \in X \times \mathbb{R}^m$ and for some $C_1 > 0$.

Theorem 1.2 *Let X be a locally compact, separable metric space and μ_n, $\mu \in [M_b(X)]^m$; if $\mu_n \overset{*}{\rightharpoonup} \mu$ then*

$$\liminf_{n\to\infty} \int_X f\left(x, \frac{d\mu_n}{d|\mu_n|}(x)\right) \, d|\mu_n| \geq \int_X f\left(x, \frac{d\mu}{d|\mu|}(x)\right) \, d|\mu|$$

for every continuous function $f : X \times \mathbb{R}^m \to \mathbb{R}$, positively 1-homogeneous and convex in the second variable, satisfying the growth condition $|f(x,z)| \leq C|z|$ for each $(x,z) \in X \times \mathbb{R}^m$ and for some $C > 0$.

Proofs to Theorems 1.1 and 1.2 have been given in [20], [19], and [5], and although the statement of the hypothesis varies, the technique is essentially the same. The idea has been to construct measures in the product space $X \times S^{m-1}$ and use a disintegration theorem to analyze the projections of the measures (see [5], Theorem 2.28). There has been some work involving arguments specific to particular problems, for example, time-dependent problems [17], as well as the desire to consider f that are not necessarily 1-homogeneous ([15], [16]). However, these arguments either use the original theorem or are applicable only to the problem they were constructed for.

It was first noticed by Luckhaus and Modica [19] that the hypothesis are overly restrictive, commenting that it was perhaps due to a translation error. An examination of the proof reveals that it is sufficient to assume the condition the condition (1.1) holds simply for the right choice of g, and in particular, one good choice is $g(x,z) = |z|$. Luckhaus and Modica make this modification in their proof in the case where $X = \Omega \subset \mathbb{R}^N$ is open, assuming

$$\lim_{n\to\infty} \int_\Omega \left|\frac{d\mu_n}{d|\mu_n|}(x)\right| \, d|\mu_n| = \int_\Omega \left|\frac{d\mu}{d|\mu|}(x)\right| \, d|\mu|,$$

which can be written in the simpler form

$$|\mu_n|(\Omega) \to |\mu|(\Omega). \tag{1.3}$$

In fact, replacing (1.1) with (1.3) in the original proof allows it to go through in the full generality of a locally compact, separable metric space. However, note that the assumption $X \subset \mathbb{R}^N$ is not as restrictive as it looks, since locally compact topological vector spaces are finite dimensional (see 1.9 in [21])[1]. Further, in the literature these theorems are usually applied in problems involving functions of bounded variation $BV(\Omega; \mathbb{R}^m)$, where $\Omega \subset \mathbb{R}^N$. In this paper we show that in the Euclidean setting it is possible to give simple proofs of Theorems 1.1 and 1.2 which have the full generality of the original theorems and do not make use of the disintegration theorem.

Theorem 1.3 *Let $\Omega \subset \mathbb{R}^N$ be open, $\mu_n, \mu \in [M_b(\Omega)]^m$ such that $\mu_n \overset{*}{\rightharpoonup} \mu$ and $|\mu_n|(\Omega) \to |\mu|(\Omega)$. Then*

$$\lim_{n\to\infty} \int_\Omega f\left(x, \frac{d\mu_n}{d|\mu_n|}(x)\right) \, d|\mu_n| = \int_\Omega f\left(x, \frac{d\mu}{d|\mu|}(x)\right) \, d|\mu|$$

for every continuous and bounded function $f : \Omega \times S^{m-1} \to \mathbb{R}$.

[1] Thus, if the metric on X comes from a norm or is compatible with the topology of a topological vector space, then X is automatically finite dimensional.

Proof. We claim it is enough to demonstrate

$$\lim_{n\to\infty} \int_{\Omega'} f\left(x, \frac{d\mu_n}{d|\mu_n|}(x)\right)\, d|\mu_n| = \int_{\Omega'} f\left(x, \frac{d\mu}{d|\mu|}(x)\right)\, d|\mu|$$

for every $\Omega' \subset\subset \Omega$ such that $|\mu|(\partial\Omega') = 0$. If this is the case, we may estimate the boundary layer by

$$\int_{\Omega\setminus\overline{\Omega'}} f\left(x, \frac{d\mu_n}{d|\mu_n|}(x)\right)\, d|\mu_n| \le M\, |\mu_n| \left(\Omega \setminus \overline{\Omega'}\right) \tag{1.4}$$

$$\int_{\Omega\setminus\overline{\Omega'}} f\left(x, \frac{d\mu}{d|\mu|}(x)\right)\, d|\mu| \le M\, |\mu| \left(\Omega \setminus \overline{\Omega'}\right), \tag{1.5}$$

where $M := \sup_{(x,z)\in\Omega\times S^{m-1}} |f(x,z)|$. Computing the limit of (1.4), we have

$$\begin{aligned}\lim_{n\to\infty} |\mu_n|(\Omega\setminus\overline{\Omega'}) &= \lim_{n\to\infty} |\mu_n|(\Omega) - \lim_{n\to\infty} |\mu_n|(\overline{\Omega'}) \\ &= |\mu|(\Omega) - |\mu|(\overline{\Omega'}) = |\mu|(\Omega\setminus\overline{\Omega'}),\end{aligned}$$

where we have used the fact that $|\mu|(\partial\Omega') = 0$ to obtain the convergence $|\mu_n|(\overline{\Omega'}) \to |\mu|(\overline{\Omega'})$ (see Proposition 1.62(b) in [5]). We can then choose Ω' appropriately to make (1.4) and (1.5) arbitrarily small. As for the interior, define $\tilde{f} : \Omega \times \overline{B(0,1)} \subset \mathbb{R}^m \to \mathbb{R}$ by

$$\tilde{f}(x,z) = \begin{cases} f(x, \frac{z}{|z|})|z| & \text{if } 0 < |z| \le 1, \\ 0 & \text{if } z = 0. \end{cases}$$

Then since f is bounded and continuous, we have that $\tilde{f}$ is bounded and continuous, and for Ω' compactly contained in Ω, $\tilde{f} : \Omega' \times \overline{B(0,1)} \to \mathbb{R}$ is uniformly continuous. Thus for every $\delta > 0$, there exists $C_\delta > 0$ such that

$$\left|\tilde{f}(x,y) - \tilde{f}(x,z)\right| \le C_\delta |y-z|^2 + \delta \tag{1.6}$$

for all $x \in \Omega'$ and $y, z \in \overline{B(0,1)}$. Let $\varphi : \Omega \to \overline{B(0,1)} \subset \mathbb{R}^m$ be continuous, to be chosen later. Then add and subtract zero to what remains in the interior to obtain

$$\begin{aligned}&\left|\int_{\Omega'} f\left(x, \frac{d\mu_n}{d|\mu_n|}(x)\right)\, d|\mu_n| - \int_{\Omega'} f\left(x, \frac{d\mu}{d|\mu|}(x)\right)\, d|\mu|\right| \\ &\le \left|\int_{\Omega'} \tilde{f}\left(x, \frac{d\mu_n}{d|\mu_n|}(x)\right)\, d|\mu_n| - \int_{\Omega'} \tilde{f}\left(x, \varphi(x)\right)\, d|\mu_n|\right| \\ &\quad + \left|\int_{\Omega'} \tilde{f}\left(x, \varphi(x)\right)\, d|\mu_n| - \int_{\Omega'} \tilde{f}\left(x, \varphi(x)\right)\, d|\mu|\right| \\ &\quad + \left|\int_{\Omega'} \tilde{f}\left(x, \varphi(x)\right)\, d|\mu| - \int_{\Omega'} \tilde{f}\left(x, \frac{d\mu}{d|\mu|}(x)\right)\, d|\mu|\right| \\ &= I + II + III.\end{aligned}$$

Let $\psi \in C_c(\Omega)$, $0 \le \psi \le 1$, $\psi = 1$ on Ω'. Considering II, we have the inequality

$$\begin{aligned}II \le &\left|\int_{\Omega} \tilde{f}\left(x, \varphi(x)\right) \psi(x)\, d|\mu_n| - \int_{\Omega} \tilde{f}\left(x, \varphi(x)\right) \psi(x)\, d|\mu|\right| \\ &+ M\left(|\mu_n|\left(\Omega\setminus\overline{\Omega'}\right) + |\mu|\left(\Omega\setminus\overline{\Omega'}\right)\right).\end{aligned}$$

The first term goes to zero as n goes to infinity by Proposition 1.80 in [5], which asserts the convergence $|\mu_n| \overset{*}{\rightharpoonup} |\mu|$, while the second is controlled by the same estimates as (1.4) and (1.5). As for I and III, by (1.6) we can bound

$$
\begin{aligned}
I + III &\leq \int_\Omega \left(C_\delta \left| \frac{d\mu_n}{d|\mu_n|}(x) - \varphi(x) \right|^2 + \delta \right) \psi(x)\, d|\mu_n| \\
&\quad + \int_\Omega \left(C_\delta \left| \frac{d\mu}{d|\mu|}(x) - \varphi(x) \right|^2 + \delta \right) \psi(x)\, d|\mu| \\
&\leq \int_\Omega \left(2C_\delta \left(1 - \frac{d\mu_n}{d|\mu_n|}(x) \cdot \varphi(x) \right) + \delta \right) \psi(x)\, d|\mu_n| \\
&\quad + \int_\Omega \left(2C_\delta \left(1 - \frac{d\mu}{d|\mu|}(x) \cdot \varphi(x) \right) + \delta \right) \psi(x)\, d|\mu|,
\end{aligned}
$$

where in the last inequality we have used that $|\varphi| \leq 1$. Letting $n \to \infty$, by the weak-star convergence $\mu_n \overset{*}{\rightharpoonup} \mu$, $|\mu_n| \overset{*}{\rightharpoonup} |\mu|$, we have

$$
\begin{aligned}
I + III &\leq 2 \int_\Omega \left(2C_\delta \left(1 - \frac{d\mu}{d|\mu|}(x) \cdot \varphi(x) \right) + \delta \right) d|\mu| \\
&= 2\delta|\mu|(\Omega) + 4C_\delta \int_\Omega \left(1 - \frac{d\mu}{d|\mu|}(x) \cdot \varphi(x) \right) d|\mu|.
\end{aligned}
$$

First choosing $\delta > 0$ small, and then choosing φ close to $\frac{d\mu}{d|\mu|}$ (since $\frac{d\mu}{d|\mu|} \in L^1(\Omega, |\mu|)$, and using the density result given by Proposition 7.9 in [8]), the result is demonstrated. ■

Next we give an alternative proof of Theorem 1.2 in the Euclidean setting. To simplify the proof, we first assume the additional hypothesis $f(x, 0) = 0$ for all $x \in \Omega$, which is true if for instance f is real-valued.

Theorem 1.4 *Let $\Omega \subset \mathbb{R}^N$ be open and μ_n, $\mu \in [M_b(\Omega)]^m$; if $\mu_n \overset{*}{\rightharpoonup} \mu$, then*

$$
\liminf_{n\to\infty} \int_\Omega f\left(x, \frac{d\mu_n}{d|\mu_n|}(x)\right) d|\mu_n| \geq \int_\Omega f\left(x, \frac{d\mu}{d|\mu|}(x)\right) d|\mu|
$$

for every lower semicontinuous function $f : \Omega \times \mathbb{R}^m \to [0, \infty]$, positively 1-homogeneous and convex in the second variable such that $f(x, 0) = 0$ for all $x \in \Omega$.

Proof. Since we have assumed $f(x, 0) = 0$, we can apply Proposition 6.42 in [13] to represent f as

$$
f(x, z) = \sup_i b_i(x) \cdot z, \tag{1.7}
$$

where $b_i : \Omega \to \mathbb{R}^m$ are bounded and continuous. Following the proofs of Theorem 5.14 and Theorem 6.54 in [13], without loss of generality we may assume that

$$
\liminf_{n\to\infty} \int_\Omega f\left(x, \frac{d\mu_n}{d|\mu_n|}\right) d|\mu_n| = \lim_{n\to\infty} \int_\Omega f\left(x, \frac{d\mu_n}{d|\mu_n|}\right) d|\mu_n| < \infty. \tag{1.8}
$$

Passing to a subsequence, there exists a positive Radon measure $\nu \in M_b(\Omega)$ such that

$$f\left(x, \frac{d\mu_n}{d|\mu_n|}\right)\, d|\mu_n| \overset{*}{\rightharpoonup} \nu$$

as $n \to \infty$. We claim it is enough to show that

$$\frac{d\nu}{d|\mu|}(x_0) \geq f\left(x_0, \frac{d\mu}{d|\mu|}(x_0)\right) \text{ for } |\mu| \text{ a.e. } x_0 \in \Omega. \tag{1.9}$$

If we can prove (1.9), then by the Radon-Nikodym theorem we can write

$$\nu = \frac{d\nu}{d|\mu|}|\mu| + \nu_s,$$

where $\nu_s \geq 0$, and we have the following inequalities

$$\begin{aligned}\lim_{n\to\infty} \int_\Omega f\left(x, \frac{d\mu_n}{d|\mu_n|}(x)\right)\, d|\mu_n| \geq \nu(\Omega) &\geq \int_\Omega \frac{d\nu}{d|\mu|}(x)\, d|\mu| \\ &\geq \int_\Omega f\left(x, \frac{d\mu}{d|\mu|}(x)\right)\, d|\mu|.\end{aligned}$$

Thus, let x_0 be a Lebesgue point of $\frac{d\mu}{d|\mu|}$ with respect to the measure $|\mu|$ such that by the Besicovitch derivation theorem we have

$$\frac{d\nu}{d|\mu|}(x_0) = \lim_{\epsilon\to 0} \frac{\nu(Q(x_0,\epsilon))}{|\mu|(Q(x_0,\epsilon))} < \infty.$$

Choosing a sequence of $\epsilon_k \to 0^+$ such that $\nu(\partial Q(x_0,\epsilon_k)) = 0$,

$$\begin{aligned}\frac{d\nu}{d|\mu|}(x_0) &= \lim_{k\to\infty} \frac{\nu(Q(x_0,\epsilon_k))}{|\mu|(Q(x_0,\epsilon_k))} \\ &= \lim_{k\to\infty}\lim_{n\to\infty} \frac{1}{|\mu|(Q(x_0,\epsilon_k))} \int_{Q(x_0,\epsilon_k)} f\left(x, \frac{d\mu_n}{d|\mu_n|}(x)\right)\, d|\mu_n| \\ &\geq \liminf_{k\to\infty}\liminf_{n\to\infty} \frac{1}{|\mu|(Q(x_0,\epsilon_k))} \int_{Q(x_0,\epsilon_k)} b_i(x)\cdot \frac{d\mu_n}{d|\mu_n|}(x)\, d|\mu_n| \\ &= \liminf_{k\to\infty} \frac{1}{|\mu|(Q(x_0,\epsilon_k))} \int_{Q(x_0,\epsilon_k)} b_i(x)\cdot \frac{d\mu}{d|\mu|}(x)\, d|\mu|,\end{aligned}$$

where we have used the weak-star convergence $\mu_n \overset{*}{\rightharpoonup} \mu$. By the continuity of b_i, for every $\eta > 0$ we have that

$$\frac{1}{|\mu|(Q(x_0,\epsilon_k))} \int_{Q(x_0,\epsilon_k)} |b_i(x) - b_i(x_0)|\, d|\mu| \leq \eta,$$

whenever k is sufficiently large. Thus, we have that

$$\begin{aligned}&\lim_{k\to\infty} \frac{1}{|\mu|(Q(x_0,\epsilon_k))} \int_{Q(x_0,\epsilon_k)} b_i(x)\cdot \frac{d\mu}{d|\mu|}(x)\, d|\mu| \\ &\qquad = \lim_{k\to\infty} \frac{1}{|\mu|(Q(x_0,\epsilon_k))} \int_{Q(x_0,\epsilon_k)} b_i(x_0)\cdot \frac{d\mu}{d|\mu|}(x)\, d|\mu| \\ &\qquad = b_i(x_0)\cdot \frac{d\mu}{d|\mu|}(x_0),\end{aligned}$$

and combining this with the above, we have

$$\frac{d\nu}{d|\mu|}(x_0) \geq b_i(x_0) \cdot \frac{d\mu}{d|\mu|}(x_0). \tag{1.10}$$

Finally, taking the supremum over i and using equation (1.7), we obtain the inequality (1.9), and the result is demonstrated. ∎

We now remove the hypothesis that $f(x,0) = 0$ for all $x \in \Omega$, with some subtle analysis of the set of $x \in \Omega$ such that $f(x) = 0$.

Theorem 1.5 *Let $\Omega \subset \mathbb{R}^N$ be open and μ_n, $\mu \in [M_b(\Omega)]^m$; if $\mu_n \overset{*}{\rightharpoonup} \mu$, then*

$$\liminf_{n\to\infty} \int_\Omega f\left(x, \frac{d\mu_n}{d|\mu_n|}(x)\right)\, d|\mu_n| \geq \int_\Omega f\left(x, \frac{d\mu}{d|\mu|}(x)\right)\, d|\mu|$$

for every lower semicontinuous function $f : \Omega \times \mathbb{R}^m \to [0,\infty]$, positively 1-homogeneous and convex in the second variable.

Proof. Define the set

$$C := \{x \in \Omega : f(x,0) = 0\},$$

and note that by lower semicontinuity of f, C is a closed set. We will show that without loss of generality, the complement of C has $|\mu|$ measure zero, which combined with a representation for f on C similar to the one used in the proof of Theorem 1.4 will yield the result. Thus, we claim that $|\mu|(\Omega \setminus C) = 0$. To see this, note that assumption (1.8) implies that for n large, say $n \geq n_0$,

$$f\left(x, \frac{d\mu_n}{d|\mu_n|}(x)\right) < \infty \text{ for } |\mu_n| \text{ a.e. } x \in \Omega.$$

Fix $n \geq n_0$ and let $x \in \Omega$ be such that $f\left(x, \frac{d\mu_n}{d|\mu_n|}(x)\right) < \infty$. Applying positive 1-homogeneity and using lower semicontinuity of f, we have that

$$0 \leq f(x,0) \leq \lim_{t\to 0^+} f\left(x, t\frac{d\mu_n}{d|\mu_n|}(x)\right) = \lim_{t\to 0^+} tf\left(x, \frac{d\mu_n}{d|\mu_n|}(x)\right) = 0.$$

Thus,

$$f(x,0) = 0 \text{ for } |\mu_n| \text{ a.e. } x \in \Omega,$$

which combined with the weak-star convergence $\mu_n \overset{*}{\rightharpoonup} \mu$ implies

$$0 \leq \int_\Omega f(x,0)\, d|\mu| \leq \liminf_{n\to\infty} \int_\Omega f(x,0)\, d|\mu_n| = 0,$$

so

$$f(x,0) = 0 \text{ for } |\mu| \text{ a.e. } x \in \Omega.$$

By Proposition 6.42 in [13], we may represent $f : C \times \mathbb{R}^m \to [0,\infty]$ as

$$f(x,z) = \sup_i b_i(x) \cdot z,$$

where $b_i : C \to \mathbb{R}^m$ are bounded and continuous. Now, since C is closed, by the Tietsze extension theorem we may extend b_i to $\tilde{b}_i : \Omega \to \mathbb{R}^m$ such that $\tilde{b}_i$ are still bounded and continuous. But then examining the blowup argument in the previous proof under these modifications, for any $x_0 \in C$ and n large we have

$$\begin{aligned}
\frac{d\nu}{d|\mu|}(x_0) &= \lim_{k\to\infty} \frac{\nu(Q(x_0,\epsilon_k))}{|\mu|(Q(x_0,\epsilon_k))} \\
&= \lim_{k\to\infty} \lim_{n\to\infty} \frac{1}{|\mu|(Q(x_0,\epsilon_k))} \int_{Q(x_0,\epsilon_k)\cap C} f\left(x, \frac{d\mu_n}{d|\mu_n|}(x)\right) \, d|\mu_n| \\
&\geq \liminf_{k\to\infty} \liminf_{n\to\infty} \frac{1}{|\mu|(Q(x_0,\epsilon_k))} \int_{Q(x_0,\epsilon_k)\cap C} \tilde{b}_i(x) \cdot \frac{d\mu_n}{d|\mu_n|}(x) \, d|\mu_n| \\
&= \liminf_{k\to\infty} \frac{1}{|\mu|(Q(x_0,\epsilon_k))} \int_{Q(x_0,\epsilon_k)} \tilde{b}_i(x) \cdot \frac{d\mu}{d|\mu|}(x) \, d|\mu|,
\end{aligned}$$

where we have used both $|\mu|$ and $|\mu_n|$ negligibilty of C for n large. However, this again says that

$$\frac{d\nu}{d|\mu|}(x_0) \geq \tilde{b}_i(x_0) \cdot \frac{d\mu}{d|\mu|}(x_0) = b_i(x_0) \cdot \frac{d\mu}{d|\mu|}(x_0),$$

since $x_0 \in C$ and $\tilde{b}_i$ is an extension of b_i. This inequality is similar to (1.10) in Theorem 1.4, and we follow the remainder of the argument of Theorem 1.4 to reach the desired conclusion. ■

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